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Algorithmen und Wahrscheinlichkeit
Theoretical Exercise 2

Submit by Moodle () until 16:00 on 20.03.2025.

Exercise 1 – *Graph coloring*

(a) Let $G = (V, E)$ be a graph, and $w \in V$ be a vertex, such that for all $v \in V \setminus \{w\}$ we have $\deg(v) \leq k - 1$. Show that G has a proper coloring into at most k colors.

(b) Let $G = (V, E)$ be a graph, and let $S \subset V$ be a set of all vertices with degree larger than $k - 1$. Show that if $|S| \leq k$, there is a proper coloring of G into at most k colors. Describe an algorithm finding such a coloring in time $O(|V| + |E|)$.

(c) Let $G = (V, E)$ be a graph with $|E| < \binom{k}{2}$. Show that there exist a proper coloring of G into k colors.

Solution 1

(a) Let us consider an order on vertices $v_1, v_2, \ldots v_n$, where $n = |V|$, such that $v_1 = w$. Let us now use the greedy algorithm which colors k in this order. Concretely, we iterate over $j \in \{1, \ldots, n\}$ and color a vertex v_j with a color in $[k]$ that is not present among colors of $N(v_j) \cap \{v_1, \ldots v_{j-1}\}$, where $N(v_j)$ is the neighborhood of the vertex v_j. Since for all $j > 1$, the $|N(v_j) \cap \{v_1, \ldots v_{j-1}\}| \leq |N(v_j) \leq k - 1$ (and obviously, when $j = 1$, we can pick any color we want), at each stage we have at least one free color in $[k]$ to use.

(b) Consider an order $v_1, v_2, \ldots v_n$ on vertices V, s.t. $\{v_1, \ldots v_{|S|}\} = S$ — we first put all vertices of degree larger than $k-1$ and then remaining vertices. We will show that for every j we have $|N(v_j) \cap \{v_1, \ldots v_{j-1}\}| \leq k-1$. Indeed, if $j \leq |S|$, then $|N(v_j) \cap \{v_1, \ldots v_{j-1}\}| \leq j-1 \leq k-1$ since $|S| \leq k$. On the other hand, if $j > |S|$, then $v_j \notin S$ and therefore $|N(v_j) \cap \{v_1, \ldots v_{j-1}\}| \leq |N(v_j)| \leq k - 1$.

Now a greedy algorithm as in the solution of (a) will iterate over $j \in \{1, \ldots n\}$ and color v_j into the first color not present among the colors of the $N(v_j) \cap \{v_1, \ldots v_{j-1}\}$. Since we showed that this set is always of size at most $k - 1$, there is always free color in $[k]$ to use.

This leads directly to an algorithm running in time $O(|V| + |E|)$: in the first stage, we iterate over all vertices, calculate a degree of every vertex and identify if it is in S or not (this takes time $O(|V| + |E|)$). Then we produce the order $v_1, \ldots v_n$ as above, by taking first all elements from S, and then all remaining vertices. Finally, we can implement the greedy coloring, for a given order in time $O(V + E)$, again by just iterating over all vertices, and for each vertex iterating over all its neighbors.

(c) Let S be a set of all vertices of degree at least k in G. We have

$$2|E| = \sum_{v \in V} \deg_G(v) \geq \sum_{v \in S} \deg_G(v) \geq k|S|.$$

Since $|E| < \frac{k(k-1)}{2}$, after we get $k|S| \leq 2|E| < k(k-1)$, hence $|S| < k - 1$, and in particular $|S| \leq k$ — we are now in the situation of item (b), and the conclusion of (b) applies.