Minitest 2 – Solutions

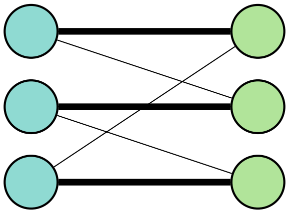
1. Every k-regular bipartite graph $G = (A \cup B, E)$ for $k \geq 1$ has a matching of size $|A|$.

Answer: TRUE

Explanation. In a k-regular bipartite graph every vertex has degree k. Since the graph is bipartite, each edge connects a vertex of A to one of B. Counting edges from both sides gives

$$k|A| = |E| = k|B|$$

so $|A| = |B|$. By Hall's Theorem such a graph has a perfect matching, which therefore matches all vertices of A (actual proof in session exercise S4).



2. The maximum matching in a bipartite graph can be found in time $O(\sqrt{|V|}|E|)$.

Answer: TRUE

Explanation. The Hopcroft–Karp (Satz 1.49 in script) algorithm finds a maximum matching in bipartite graphs in time

$$O(\sqrt{|V|}\,|E|)$$

by repeatedly finding multiple shortest augmenting paths simultaneously.

3. Every graph without a triangle has chromatic number at most 100.

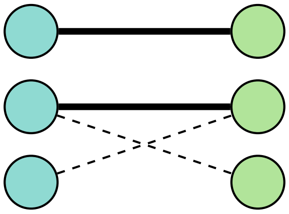
Answer: FALSE

Explanation. There exist triangle-free graphs with arbitrarily large chromatic number (Satz 1.66 in script, Mycielski constructions)

4. If G is bipartite and M is not a maximum matching, then there exists an augmenting path.

Answer: TRUE

Explanation. By **Satz 1.48 in script: Satz von Berge**, a matching M is maximum iff there is no augmenting path. Therefore if M is not maximum, an augmenting path must exist.



The dashed alternating path increases the matching size.

5. If G is connected with maximum degree 100, then G has a proper coloring with 100 colors unless it is complete.

Answer: TRUE

Explanation. This follows from **Brooks' Theorem (Satz 1.64 in script)**. It states that every connected graph with maximum degree Δ has a Δ-coloring unless it is

- a complete graph, or
- an odd cycle.

Since odd cycles have $\Delta = 2$, they are irrelevant for $\Delta = 100$.

6. For a complete graph with metric weights, the minimum perfect matching has cost $\leq \ell(C)/2$.

Answer: TRUE

Explanation. Take an optimal TSP tour C. Removing every second edge yields a perfect matching. The resulting matching has cost at most half the cost of C.

7. A greedy algorithm uses at most $\Delta(G) + 1$ colors.

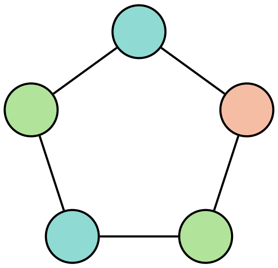
Answer: TRUE

Explanation. Satz 1.60 in script

8. Every cycle has a proper 2-coloring.

Answer: FALSE

Explanation. Only **even cycles** are bipartite and therefore 2-colorable. Odd cycles require three colors.



The odd cycle (e.g., C_5) requires a third color because the final vertex is adjacent to both existing colors.

9. There is a polynomial-time 1.5-approximation for Metric TSP.

Answer: TRUE

Explanation. **Christofides' algorithm** Satz 1.51 in script

10. There is a polynomial-time algorithm finding a proper 6-coloring for every planar graph.

Answer: TRUE

Explanation. Planar graphs are known to be 6-colorable via a simple greedy argument. Intuitively, planar graphs are "sparse." Because edges cannot cross, there is a limit to how many edges a graph can have, which ensures that there is always at least one vertex of degree ≤ 5 (more formal in lecture).

We apply **Satz 1.65** from the script:

> *If every induced subgraph of G contains a vertex with degree $\leq k$, then $\chi(G) \leq k+1$ and a $(k+1)$-coloring can be found in $O(|E|)$ time.*

Since any induced subgraph of a planar graph is also planar, every subgraph will always contain a vertex of degree ≤ 5. Setting $k = 5$, it follows that $\chi(G) \leq 5 + 1 = 6$. Repeatedly removing such vertices and coloring backwards yields a 6-coloring.