Minitest 3 – Solutions

1. Three events A, B, C are independent if and only if $\Pr[A \cap B \cap C] = \Pr[A]\Pr[B]\Pr[C]$.

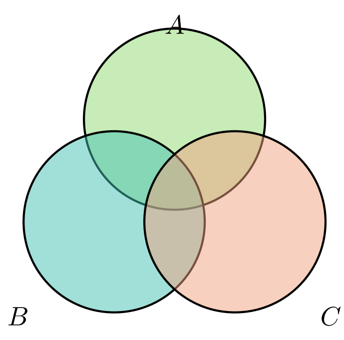
Answer: FALSE

Explanation. Total independence for three events requires both the triple product **and** pairwise independence for all combinations (e.g., $\Pr[A \cap B] = \Pr[A]\Pr[B]$)[Definition 2.22. in Script]. The triple product alone is insufficient to guarantee that the events do not influence each other in pairs.

2. For any three events A, B, C, we have $\Pr[A \cup B \cup C] = \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C] + \Pr[A \cap B \cap C]$.

Answer: TRUE

Explanation. This is the standard **Inclusion-Exclusion Principle** (Siebformel) for three events[Satz 1.35. in Script]. It correctly calculates the union by adding individual probabilities, subtracting double-counted overlaps, and adding back the triple intersection that was removed entirely during the subtraction phase.



3. We consider the following random experiment: First, we roll a six-sided die and then flip a coin. This random experiment can be described by the sample space $\Omega = \{1, 2, 3, 4, 5, 6, H, T\}$.

Answer: FALSE

Explanation. The sample space must represent the combined outcomes of both steps as **ordered pairs**. The correct sample space is the Cartesian product $\Omega = \{1, 2, 3, 4, 5, 6\} \times \{H, T\}$, which contains 12 elements such as $(1, H), (1, T), \ldots, (6, T)$.

4. For any two events A, B with $\Pr[A] > 0$ and $\Pr[B] > 0$ we have $\Pr[A|B] \Pr[B] = \Pr[B|A] \Pr[A]$.

Answer: TRUE

Explanation. This identity follows directly from the definition of conditional probability[Definition 2.8.]. Both sides of the equation are mathematically equal to the probability of the intersection, $\Pr[A \cap B]$.

5. You roll a six-sided die. Then the events $A =$ "the result is even" and $B =$ "the result is 6" are independent.

Answer: FALSE

Explanation. Two events are independent only if $\Pr[A|B] = \Pr[A]$ (Given $\Pr[B] > 0$) [Definition 2.8. and Definition 2.18.]. Here, $\Pr[A] = 1/2$ (for results 2, 4, 6), but if we know B has occurred (the result is 6), then the result is definitely even, so $\Pr[A|B] = 1$. Since $1/2 \neq 1$, the events are dependent.

6. If A, B, C are three events satisfying $\Pr[A \cap B] = \Pr[A] \Pr[B]$, $\Pr[A \cap C] = \Pr[A] \Pr[C]$, and $\Pr[B \cap C] = \Pr[B] \Pr[C]$, then A, B, C are independent.

Answer: FALSE

Explanation. This condition only establishes **pairwise independence**. As noted in Question 1, total independence for three events also requires the triple product rule $\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C]$ to hold[Definition 2.22.].

7. If A, B, C are three independent events, then $\Pr[A \cap B] = \Pr[A]\Pr[B]$, $\Pr[B \cap C] = \Pr[B]\Pr[C]$, and $\Pr[A \cap C] = \Pr[A]\Pr[C]$.

Answer: TRUE

Explanation. By definition, if a set of events is independent, all possible subsets of those events (including all pairs) must satisfy the product rule[Definition 2.22.].

8. We have 180 random people in a room. The probability that two of them share the same birthday is less than $1/2$.

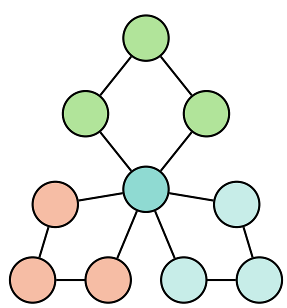
Answer: FALSE

Explanation. According to the **Birthday Paradox**, the probability of a shared birthday exceeds 0.5 once there are just 23 people in a room [Wikipedia: Birthday problem]. With 180 people, the probability is extremely close to 1.

9. Suppose G is a graph that contains an Euler tour, and the number of vertices of G is even. Then G contains a perfect matching.

Answer: FALSE

Explanation. Counterexample:



10. If A and B are independent events then $\Pr[A \cup B] = \Pr[A] + \Pr[B]$.

Answer: FALSE

Explanation. The additive rule $\Pr[A \cup B] = \Pr[A] + \Pr[B]$ only holds if the events are **disjoint** ($\Pr[A \cap B] = 0$)[(Siebformel) Satz 1.35. in Script]. If two events with positive probabilities are independent, their intersection $\Pr[A] \cdot \Pr[B]$ must be positive, meaning they cannot be disjoint (since $\Pr[A \cap B] > 0$).