Minitest 4 – Solutions

1. For any random variables $X_1, X_2, \ldots, X_n$ and constants $a_1, \ldots, a_n$, we have $\mathbb{E}[a_1X_1 + a_2X_2 + \cdots + a_nX_n] = a_1\mathbb{E}[X_1] + \cdots + a_n\mathbb{E}[X_n]$.

Answer: TRUE

Explanation. This is the **Linearity of Expectation** [Satz 2.33. in Script].

2. For any independent random variables X_1, X_2 we have $\mathbf{Var}(X_1 + X_2) = \mathbf{Var}(X_1) + \mathbf{Var}(X_2)$.

Answer: TRUE

Explanation. If the variables are **independent**, Variance is additive [Satz 2.62.].

3. Let X, Y be independent random variables with $\mathbf{Var}(X) = 1$ and $\mathbf{Var}(Y) = 4$. Then $\mathbf{Var}(X - Y) = -3$.

Answer: FALSE

Explanation. Variance measures "spread," so it must be **non-negative**. When you subtract a variable, you are still adding to the uncertainty (spread).

$$\mathrm{Var}(X - Y) = \mathrm{Var}(X) + (-1)^2\mathrm{Var}(Y) = 1 + 4 = 5$$

4. If X and Y are independent random variables then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

Answer: TRUE

Explanation. While expectation always splits across sums (linearity), if the variables are independent, it also splits across products [Satz 2.61.].

5. If $X_1, X_2, \ldots, X_n$ are random variables such that $\mathrm{Var}(X_i) = 1$ for all i, then $\mathrm{Var}(X_1 + \cdots + X_n) = n$.

Answer: FALSE

Explanation. The statement fails if the variables are **dependent**. Counterexample: For $X_1 = X_2 = \cdots = X_n$, the variance would be n^2 instead of n, since $\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(nX_1) = n^2\mathrm{Var}(X_1) = n^2$.

6. For a random variable X with $\mathrm{Var}(X) = \sigma^2$ and $\mathbb{E}[X] = 0$, and any $\lambda > 0$, we have: $\Pr(X > \lambda\sigma) \leq 1/\lambda^2$.

Answer: TRUE

Explanation. This is an application of the **Chebyshev Inequality** [Satz 2.68.]. Since $\Pr(X > \lambda\sigma) \leq \Pr(|X| \geq \lambda\sigma)$, and the latter is capped at $1/\lambda^2$, the statement holds.

7. Let X, Y, Z be three random variables such that X and Y are independent. Then it always holds that $\mathbb{E}[X + Y \cdot Z] = \mathbb{E}[X] + \mathbb{E}[Y] \cdot \mathbb{E}[Z]$.

Answer: FALSE

Explanation. By linearity, $\mathbb{E}[X+YZ] = \mathbb{E}[X]+\mathbb{E}[YZ]$. However, $\mathbb{E}[YZ] = \mathbb{E}[Y]\mathbb{E}[Z]$ is only guaranteed if Y and Z are independent. The statement only guarantees that X and Y are independent, but Y and Z could be dependent.

8. If X is a non-negative random variable with $\mathbb{E}[X] > 100$, then $\Pr[X > 10] \geq 1/2$.

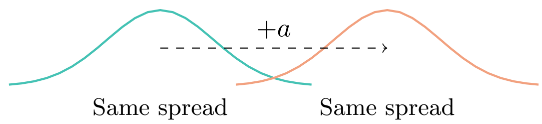
Answer: FALSE

Counterexample: Let $X = 1001$ with probability 0.1 and 0 otherwise. $\mathbb{E}[X] = 100.1 > 100$, but $\Pr[X > 10] = 0.1$, which is much smaller than 1/2.

9. For a random variable X and a constant a, we have $\mathbf{Var}(X + a) = \mathbf{Var}(X) + a$.

Answer: FALSE

Explanation. Intuition: Adding a constant shifts the distribution but does not change its shape or spread. $\text{Var}(X + a) = \text{Var}(X)$ [Satz 2.41.]



10. If we throw independently at random n balls into n bins (with each ball landing in each bin with equal probability), and X_1 denotes the number of balls in the first bin at the end of the process, then $\mathbb{E}[X_1] = 1$.

Answer: TRUE

Explanation. Each ball has a $1/n$ probability of landing in Bin 1. With n balls, the expected number is $n \times \frac{1}{n} = 1$.