S5

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Exercise S5.1 – ***3-colorable graphs***

In the lecture you have seen an algorithm that finds an $O\left(\sqrt{|V|}\right)$-coloring for every 3-colorable graph (V, E). Fix $\boxed{\alpha = \sqrt{|V|}}$. The algorithm goes as follows.

1. While there is a vertex v with degree at least α, color v with a new color, color the neighbors of v with two additional new colors, and delete v and all its neighbors from the graph.
2. Color all remaining vertices with at most $\alpha + 1$ colors.

We assume $\{v\} \cup N(v)$ is tripartite

$\Rightarrow \Delta(\text{Rest}) \leq \alpha$

We want to analyze and generalize this algorithm.

(a) Where in the algorithm do we use that the graph is 3-colorable?

(b) How many colors do we need at most (depending on α)? Show that for $\alpha = \sqrt{|V|}$ we only need $O\left(\sqrt{|V|}\right)$ colors.

$= O(\alpha)$

(c) Show that your bound in (a) is tight. More precisely, create a 3-colorable graph on which the algorithm uses $\Omega\left(\sqrt{|V|}\right)$ colors.

a) ——— $|V| = n$

b) Let k be # of iterations

We need 3 colours per iteration of loop

$\Rightarrow$ for ① I need $3k$ colours

Per iteration we delete $\geq \alpha + 1$ vertices

$$\Rightarrow k \leq \frac{n}{\alpha+1} \leq \frac{n}{\alpha} = \alpha \qquad \left(\frac{|V|}{\sqrt{|V|}} = \sqrt{|V|}\right)$$

In ② we colour the rest in $\alpha + 1$ colours (Greedy alg.)

In total we use $\leq 3\alpha + \alpha + 1 = 4\sqrt{|V|} + 1$ colours

and $4\sqrt{|V|} + 1 \in O(\sqrt{|V|})$

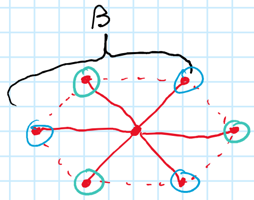
c)

(c) Fix sone constant β. A wheel of size 2β is a cycle of length 2β and an additional "center vertex" that has one edge to each vertex on the cycle. Note that every wheel is 3-colorable. Let G be a graph consisting of β many disjoint wheels of size 2β. The graph G has $\beta(2\beta+1) = |V| \Rightarrow O(\sqrt{|V|}) = O(\beta)$ many vertices. Each center vertex has a degree of $2\beta > \sqrt{\beta(2\beta+1)}$. Hence, when applied to G, the algorithm will execute β many iterations of the while loop ① (one for each "center vertex"). In each iteration, it introduces 3 new colors. Hence, it uses at $3\beta = \Omega\left(\sqrt{|V|}\right)$ colors.

$= \Omega(\beta)$

wts. $\geq \Omega(\beta)$ many colours

β



3 colours

β many of those

(d) Can you describe an algorithm that finds a $O\left(|V|^{\frac{2}{3}}\right)$-coloring for every 4 colorable graph?

(e) Can you describe an algorithm that finds a $O\left(|V|^{\frac{q-2}{q-1}}\right)$-coloring for every q colorable graph, where $q \geq 2$ is a constant?

d) Set $\alpha = |V|^{2/3}$

1. While there is a vertex v with degree at least α, color v with a new color, color the neighbors of v with ~~two~~ $4\sqrt{\alpha}+1$ using the algorithm from before additional new colors, and delete v and all its neighbors from the graph.
2. Color all remaining vertices with at most $\alpha + 1$ colors. with Greedy alg.

$k \leq \frac{n}{\alpha+1} \leq \frac{n}{\alpha}$

colours used:

1. $k(4\sqrt{\alpha}+1)$
2. $\alpha + 1$

$$\text{Total: } ① + ② \leq \frac{n}{\alpha}(4\sqrt{\alpha}+1) + \alpha + 1 = \overbrace{\frac{|V|}{|V|^{2/3}}}^{|V|^{1/3}} \cdot \left(4\overbrace{\left(|V|^{2/3}\right)^{1/2}}^{|V|^{1/3}}+1\right) + |V|^{2/3} + 1$$

for $\alpha = |V|^{2/3} \Rightarrow$ at most $O(|V|^{2/3})$ colours