S6

26 March 2026 13:20

Exercise S6.1 – ***Probability Space***

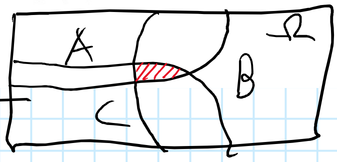
(a) For each of the following subtasks, either define a probability space and events A and B (and C) with the described properties, or prove that such a space cannot exist. Make sure that you define both, the sample space ("Ergebnismenge") Ω and the probabilities of the atomic events ("Elementarereignisse").

(i) $\Pr[A] = \frac{1}{4}$, $\Pr[B] = \frac{1}{3}$ and $\Pr[A \cup B] = \Pr[A] + \Pr[B]$. $\Longrightarrow \Pr[A \cap B] = 0$

(ii) $\Pr[A] = \frac{1}{4}$, $\Pr[B] = \frac{1}{3}$ and $\Pr[A \cup B] < \Pr[A] + \Pr[B]$.

(iii) $\Pr[A] = \Pr[B]$, $\Pr[A \cap B] = \frac{1}{4}$, and $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$ (that is A and B are independent). $\Rightarrow$ bspw. $\Pr[A] = 1/2 = \Pr[B]$

(iv) $\Pr[A] = \Pr[B] = \Pr[C] = \frac{5}{6}$ and $\Pr[A \cap B \cap C] = 0$.



can this be ?

$\Omega := \{1, 2, 3, \dots, 11, 12\}$ 12 sided die

and $\forall \omega \in \Omega$ $\Pr[\omega] = \frac{1}{12}$ (Laplacian space)

(i) $A := \{1, 2, 3\}$, $B := \{9, 10, 11, 12\}$

$\Rightarrow \Pr[A] = 3/12 = 1/4$, $\Pr[B] = 4/12 = 1/3$

$\Pr[A \cup B] = \Pr[\{1, 2, 3, 9, 10, 11, 12\}] = 7/12 = \Pr[A] + \Pr[B]$

(ii) —

(iii) $\Omega := \{1, 2, 3, 4\}$ $\forall \omega \in \Omega$ $\Pr[\omega] = 1/4$

$A := \{1, 2\}$ $B := \{2, 4\}$

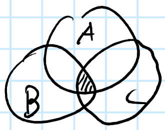
$\Rightarrow \Pr[A] = 1/2 = \Pr[B]$, $\Pr[A \cap B] = \Pr[\{2\}] = 1/4 = \Pr[A] \cdot \Pr[B]$

(iv) No Ω exists:

Assume Ω exists s.t. $\Pr[A] = \Pr[B] = \Pr[C] = 5/6$ and $\Pr[A \cap B \cap C] = 0$

Note $\Pr[\bar{A}] = 1 - \Pr[A]$

$$1 = 1 - \Pr[A \cap B \cap C]$$



$$= \Pr[\bar{A} \cup \bar{B} \cup \bar{C}]$$

$$\overset{\text{U.B.}}{\leq} \Pr[\bar{A}] + \Pr[\bar{B}] + \Pr[\bar{C}] = 3/6 = 1/2 < 1$$

$\Rightarrow 1 < 1 \Rightarrow$ Contradiction