S7

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Exercise S7.1 – *Intransitive Dice*

Alice, Bob, and Clara just found three fair dice. Instead of the numbers 1 to 6, the six sides of the dice show the following numbers:

9999 9999 $\to \mathbb{E}[X_1]$, $\mathrm{Var}[X_1]$ much larger

Die 1: 2, 2, 4, 4, 9, 9

Die 2: 1, 1, 6, 6, 8, 8

Die 3: 3, 3, 5, 5, 7, 7

$X_3 > X_2$: (3,2), (5,2), (7,2), (3,4), (5,4), (7,4)
(3,9), (5,9), (7,9)

Bob and Clara want to play a game: Bob starts by choosing a die. Next, Clara chooses a die. Then, both of them roll the dice and the person whose die shows the higher number wins.

Alice computes the expected value of the outcome of each die. She claims that their choices do not matter, because all dice have the same expected value.

Bob additionally computes the variance for the outcome of each die. He claims that the first two dice have a higher variance and are thus better than the third die. Thus, he chooses die 1.

Clara chooses die 3 and claims that her probability of winning was $\frac{5}{9}$ all along — independent of Bob's decision.

$\Pr[X_3 > X_1] = 5/9$

Compute the expected value and variance for each die. What do you think about the claims of Alice, Bob, and Clara?

X_1, X_2, X_3 represent the outcomes of rolling die 1, 2, 3, resp.

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\begin{aligned}
\mathbb{E}[X_1] &= \frac{1}{3} \cdot (2+4+9) = 5\\
\mathbb{E}[X_2] &= \frac{1}{3} \cdot (1+6+8) = 5\\
\mathbb{E}[X_3] &= \frac{1}{3} \cdot (3+5+7) = 5\\
\mathrm{Var}[X_1] &= \frac{1}{3} \cdot ((2-5)^2 + (4-5)^2 + (9-5)^2) = \frac{26}{3}\\
\mathrm{Var}[X_2] &= \frac{1}{3} \cdot ((1-5)^2 + (6-5)^2 + (8-5)^2) = \frac{26}{3}\\
\mathrm{Var}[X_3] &= \frac{1}{3} \cdot ((3-5)^2 + (5-5)^2 + (7-5)^2) = \frac{8}{3}.
\end{aligned}
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