S8

13 April 2026 15:20

Exercise S8.1 – ***Inequalities***

We throw 1000 fair coins and denote the results by $C_1, \ldots, C_{1000}$. We want to count the number of neighbouring coins that both show "heads". Two neighbouring coins are coins of the form C_i, C_{i+1} or C_{1000}, C_1 (imagine the coins being placed on a circle). Let X denote the number of neighbouring coins that both show "heads". We will use different inequalities to bound the probability that X is significantly larger than its expected value. Make sure to check all relevant conditions before applying an inequality.

(a) Show that $\mathbb{E}[X] = 250$.

(b) Use Markov's inequality to bound $\Pr[X \geq 300]$.

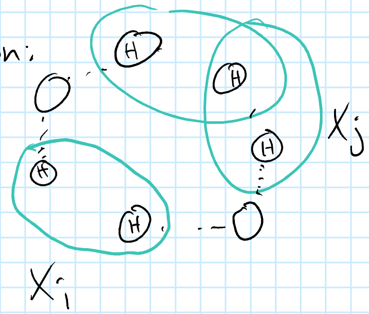
(c) Compute $\mathrm{Var}[X]$ and use Chebychev's inequality to bound $\Pr[X \geq 300]$.

$n := 1000$, When index is $1000+1$ or $1000+2$ we consider index 1 and 2 resp.

a) For $i \in [n]$ X_i is 1 iff. C_i and C_{i+1} show heads and 0 otherwise

$X = \sum_{i=1}^{n} X_i$

Situation:



$$\Rightarrow \mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \Pr[X_i = 1] = n \cdot 1/4 = 250$$

b) $\Pr[X \geq 300] \leq ?$

$$\Pr[X \geq 300] \leq \frac{\mathbb{E}[X]}{300} = \frac{250}{300} = \frac{5}{6}$$

c) Compute $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$X^2 = \left(\sum_{i=1}^{n} X_i\right)^2 = \sum_{(i,j) \in [n]^2} X_i X_j$$

Consider $S_0, S_1, S_{\geq 2}$

s.t. pairs (i,j) are in S_0 if $|i-j| = 0$,

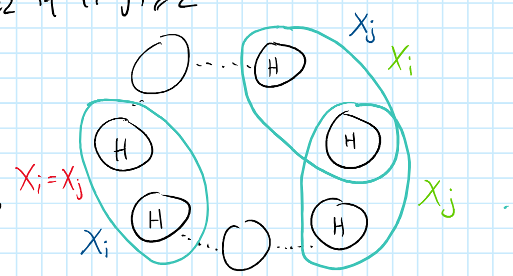
S_1 if $|i-j| = 1$,

$S_{\geq 2}$ if $|i-j| \geq 2$

For $(i,j) \in S_0$: $\Pr[X_i X_j = 1] = 1/4$

$(i,j) \in S_1$: " $= 1/8$

$(i,j) \in S_{\geq 2}$: " $= 1/16$



$$\mathbb{E}[X^2] = \sum_{(i,j) \in [n]^2} \mathbb{E}[X_i X_j]$$

$$= \sum_{(i,j) \in [n]^2} \Pr[X_i X_j = 1]$$

$$= \sum_{(i,j) \in S_0} 1/4 + \sum_{(i,j) \in S_1} 1/8 + \sum_{(i,j) \in S_{\geq 2}} 1/16$$

$$= |S_0| \, 1/4 + |S_1| \, 1/8 + |S_{\geq 2}| \cdot 1/16$$

$$(i,j)\in S_0 \qquad (i,j)\in S_1 \qquad (i,j)\in S_{\geq 2}$$

$$= |S_0|\, ^1/_4 + |S_1|\, ^1/_8 + |S_{\geq 2}| \cdot {}^1/_{16}$$

$$(|S_0| = n,\ |S_1| = 2n,\ |S_{\geq 2}| = n^2 - n - 2n = n^2 - 3n)$$

$$= \underbrace{{}^n/_4 + {}^{2n}/_8 + {}^{(n^2-3n)}/_{16}}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \downarrow - {}^{n^2}/_{16} = \frac{5n}{16}$$

$$\Pr[X \geq 300] \leq \Pr[|X - \mathbb{E}[X]| \geq 50]$$

$$\leq \frac{\mathrm{Var}[X]}{50^2}$$

$$= \frac{5000/16}{2500} = \frac{2}{16} = \frac{1}{8}$$

$$\Rightarrow \Pr[X \geq 300] \in [0, {}^1/_8]$$

(d) We define Y as the number of neighbouring coins that both show "heads" and for which the first coin has an odd index. That is, we only consider pairs of the form X_{2i-1}, X_{2i}, with $i = 1, \ldots, 500$. Show that $\mathbb{E}[Y] = 125$ and use Chernoff's bound for $\Pr[Y \geq 150]$.

(e) Use (d) to bound $\Pr[X \geq 300]$.
Remark: Could we just apply Chernoff's bound right away?