S10

Exercise S10.1 – *Probabilistic method*

So far, we mainly used randomization to construct efficient algorithms. However, randomization can also be used to prove statements that don't involve randomness themselves. In fact, such proves can usually be phrased by describing a Monte-Carlo algorithm. Answer the two following questions. If you want to, you can describe your solution as a Monte-Carlo algorithm.

(a) You are given 1000 subsets $A_1, \ldots, A_{1000}$ of $\{1, \ldots, n\}$. Each subset has size at least 11. Can the numbers $1, \ldots, n$ be colored 'red' and 'blue', such that each set A_i contain at least one 'red' and at least one 'blue' number?

We assign blue/red independ. with prob. $\frac{1}{2}$ to all numbers in $[n]$

WTS $\Pr[\forall i \in [1000].\ A_i$ has both red and blue$] > 0$

$\iff 1 - \Pr[\geq 1$ of all A_i is unicolor$] > 0$

$\Pr[A_i$ is unicolor$] = \frac{1}{2^{|A_i|}} + \frac{1}{2^{|A_i|}} = \Pr[A_i$ is fully blue$] + \Pr[A_i$ is fully red$]$

arbitrary i

$\leq 2\frac{1}{2^{11}} = \frac{1}{2^{10}}$

$\stackrel{\cup}{\Rightarrow} \Pr[\geq 1$ of all A_i is unicolor$] \leq \sum_{i=1}^{1000} \Pr[A_i$ is unicolor$] \leq \frac{1000}{2^{10}} < 1$

$\Rightarrow 1 - \Pr[\geq 1$ of all A_i is unicolor$] > 0$

$\Rightarrow \Pr[\forall i \in [1000].\ A_i$ has both red and blue$] > 0$

(b) You are given 10 points in the plane (i.e., $\mathbb{R}^2$). Can you place (filled) disks of radius 1 in the plane, such that (i) each of the 10 points is contained in a disk, and (ii) all disks are disjoint (i.e., their centers have distance at least 2)?
Remark: You may cover multiple points with the same disk. In particular, the task would be trivial if all points are very close to each other (then you only need one disk to cover all of them). Similarly, if all points are very far apart, you could use one disk per point without having to worry about disjointness.
Remark: a formal solution to this exercise would be very technical. Focus more on the ideas than on technical details

(b) We make our task even harder: Assume that the ten points are fixed, but they are not told to you. The best thing you could do in this setting would be trying to cover as much space as possible. So first, let us think about what fraction of $\mathbb{R}^2$ we can cover with disjoint unit disks (this is already a bit imprecise because this fraction would be $\frac{\infty}{\infty}$). There is a straight forward way to pack disks very dense into the plane. In particular, we can cover a

$$\frac{\pi/2}{\frac{1}{2}\cdot 2\cdot\sqrt{2^2-1^2}} = \frac{\pi\sqrt{3}}{6} > 0.906$$

fraction of the plane. (This can e.g. be seen, by covering the plane with equilateral triangles of side length 2. When putting a unit disk around each corner of the triangles, then each in each triangle, the covered area is $\pi/2$, while the size of the triangle is $\frac{1}{2}\cdot 2\cdot\sqrt{2^2-1^2}$.)

We shift this optimal coverage randomly across plane.
For $i \in [10]$ X_i indicates whether point i is covered.

$$\Pr[X_i = 1] = \frac{\pi\sqrt{3}}{6} > 0.9, \quad \Pr[X_i = 1] = \mathbb{E}[X_i] \qquad (1)$$

WTS $\Pr\left[\sum_{i=1}^{10} X_i = 10\right] > 0$

$$\mathbb{E}\left[\sum_{i=1}^{10} X_i\right] > 10 \cdot 0.9 = 9 \qquad \text{(by Linearity, (1))}$$

$\Rightarrow \Pr\left[\sum_{i=1}^{10} X_i = 10\right] > 0$, since $\sum_{i=1}^{10} X_i$ can only take values in $[10]$