

Exercise S12.1 – Integrality and Matching

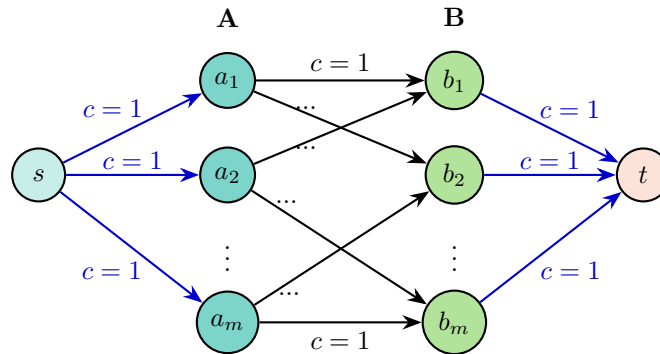
We have already seen Frobenius' Theorem: *Let $k \geq 1$. Every k -regular bipartite graph contains a perfect matching.* The proof we have seen uses Hall's theorem. In this exercise, we want to find another proof using our knowledge about flows.

Let $G = (A \cup B, E)$ be a k -regular graph for $k \geq 1$.

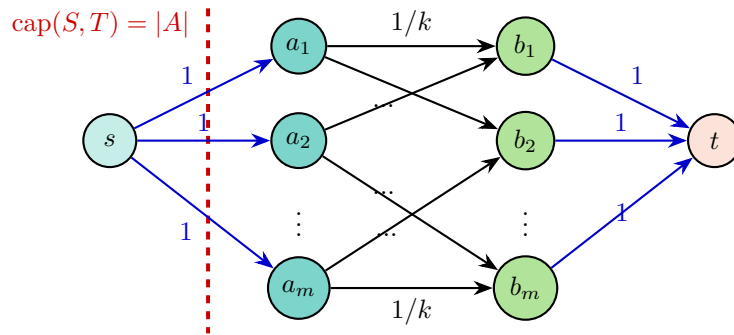
- Describe how to model bipartite matching on G as a flow problem in a network $N = (V, A, c, s, t)$.
- Construct explicitly a (not necessarily integer!) flow f on N with $\text{val}(f) = n$. Conclude that $\text{maxflow}(N) = n$.
Hint: you don't have to construct an integral flow. Make sure to define the flow value on all edges of your network N !
- Using the result from (b), show that there exists an integer valued flow f' with $\text{val}(f') = n$. Using f' , prove that G contains a perfect matching M .

Solution S12.1 – Integrality and Matching

- We do the same construction as always: we direct all edges from A to B . Furthermore, we add a source s , and all edges (s, a) for $a \in A$, and we add a sink t and all edges (b, t) for $b \in B$. Each edge gets capacity 1. There is a one to one correspondence between matchings in G and *integral* flows of value $|A| = |B|$ in N . (See Lemma 3.15.)

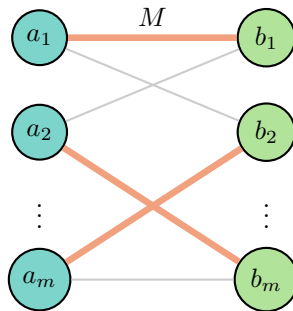


- For our flow f , we set $f(e) = 1/k$ for all edges between A and B . For the remaining edges, we set the flow to 1. Because all capacities are 1, they are satisfied. Flow conservation is also satisfied: each vertex in A has one incoming edge with flow value 1 and k outgoing edges with flow value $1/k$ each. For vertices in B it is exactly the other way around. Here we used that G is k -regular. The value of this flow equals $|A| = |B|$. The cut $\text{cap}(\{s\}, U \cup W \cup \{t\})$ has capacity $|A|$. Thus, f is a maximal flow.



Each vertex in A has k outgoing edges: $\sum f_{\text{out}} = k \cdot (1/k) = 1$.
 Each vertex in B has k incoming edges: $\sum f_{\text{in}} = k \cdot (1/k) = 1$.

- (c) By Theorem 3.12, the maximum flow in N and the maximum integral flow in N have the same value. Thus, by (b), there is an integral flow of value $|A|$. By Lemma 3.15, G has a matching of size $|A| = |B|$. This is already a perfect matching.



Perfect Matching M Discovered