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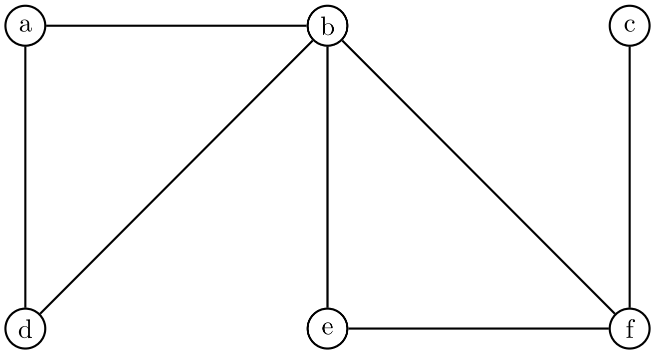
Algorithms and Probability **Session Sheet S2** **FS 2026**

For the exercise sessions on 26 February 2026.

Exercise S2.1 – *Efficient Search for Brigdes and Cut-Vertices*

In the lecture we have seen how to find bridges (Brücken) and cut-vertices (Artikulationsknoten) in a graph. The algorithm uses a modified DFS and computes values `dfs[v]` and `low[v]` for ever vertex v.

Consider the following graph G.



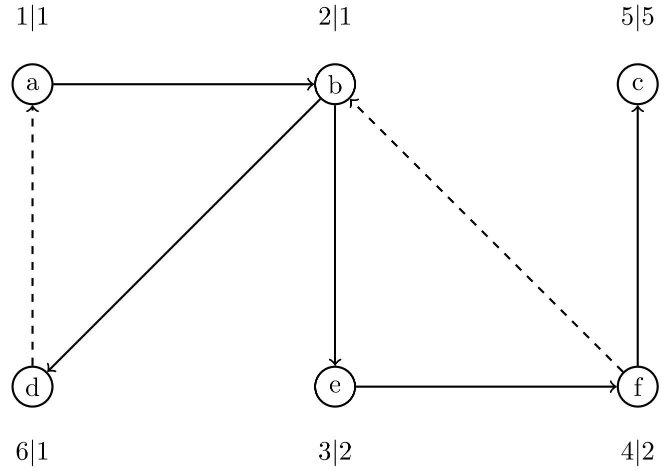
(a) Perform the modified DFS starting at vertex a. Use the corresponding `dfs`- and `low`-values to determine the cut-vertices and bridges of G.

(b) Try the same analysis, but starting in b. Do the `dfs`- and `low`-values change? Do the cut-vertices and bridges change?

(c) Compute the block graph of G.

Solution S2.1 – *Efficient Search for Brigdes and Cut-Vertices*

(a) If at each vertex you process the children in alphabetic order, the result looks as follows (`dfs`|`low`):



There are many possibilities for assigning the dfs-values, but regardless they should satisfy $low[a] = low[b] = low[d] = dfs[a]$, $low[e] = low[f] = dfs[b]$ and $low[c] = dfs[c]$. As the root, a, has one child in the DFS tree (either b or d, depending on which the DFS visited first), it is not a cut-vertex. Following the conditions for cut-vertices and cut-edges for non-root vertices, we conclude that b and f are cut-vertices and $\{c, f\}$ is a cut-edge.

(b) Now all the low-values of all vertices except c are equal to $dfs[b] = 1$. The root b is a cut-vertex as it has two children in the DFS tree. By the same conditions as before, f is a cut-vertex, and $\{c, f\}$ is a cut-edge.

We see that the cut-vertices and cut-edges are the same as in (a), as expected as these are properties of the graph and not how we set up the search.

(c) There are three blocks: $A = \{\{a, b\}, \{a, d\}, \{b, d\}\}$, $B = \{\{b, e\}, \{b, f\}, \{e, f\}\}$ and $C = \{\{c, f\}\}$. Together with the cut-vertices b and f, the block graph has vertex set $\{b, f, A, B, C\}$ and edges given by $b \sim A$, $b \sim B$, $f \sim B$ and $f \sim C$. That is, the block graph is a path with 5 vertices.

Exercise S2.2 – *Connectivity*

Let $G = (V, E)$ be a two connected graph.

(a) Let (u, v, w) be a path in G of length 2. Show that we can extend this path to a cycle, that is, show that G contains a cycle where u, v, w appear as incident vertices.

(b) Let $e \in E$ be an edge, and $u, v \in V$ be two vertices. Show that there is a path in G from u to v passing through e.
(*Hint:* you may use without proof that in a 2-connected graph, every two edges e, f lie on a common cycle.)

Solution S2.2 – *Connectivity*

(a) Let (u, v, w) be a path in G of length 2. By the 2-connectedness of G, the graph $G[V \setminus v]$ is connected. Hence, there is a u-w path P that does not contain v. Combining P and (u, v, w) gives the desired cycle.

(b) Let $e \in E$ be an edge, and $u, v \in V$ be two vertices. If $e = \{u, v\}$, the statement trivially holds (as e alone forms an u-v path). Otherwise, consider the graph $G' = (V, E \cup \{\{u, v\}\})$ that arises from G by adding the edge $\{u, v\}$ (if it did not exist already anyways). The graph

G' is also 2-connected. Applying the hint to the edges e and $\{u, v\}$, we obtain a cycle C containing both e and $\{u, v\}$. Removing $\{u, v\}$ from this cycle gives the desired path.