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Algorithms and Probability **Session Sheet S7** **FS 2026**

For the exercise sessions on 02 April 2026.

Exercise S7.1 – *Intransitive Dice*

Alice, Bob, and Clara just found three fair dice. Instead of the numbers 1 to 6, the six sides of the dice show the following numbers:

Die 1: $2, 2, 4, 4, 9, 9$

Die 2: $1, 1, 6, 6, 8, 8$

Die 3: $3, 3, 5, 5, 7, 7$

Bob and Clara want to play a game: Bob starts by choosing a die. Next, Clara chooses a die. Then, both of them roll the dice and the person whose die shows the higher number wins.

Alice computes the expected value of the outcome of each die. She claims that their choices do not matter, because all dice have the same expected value.

Bob additionally computes the variance for the outcome of each die. He claims that the first two dice have a higher variance and are thus better than the third die. Thus, he chooses die 1.

Clara chooses die 3 and claims that her probability of winning was $\frac{5}{9}$ all along — no matter which die Bob chose.

Compute the expected value and variance for each die. What do you think about the claims of Alice, Bob, and Clara?

Solution S7.1 – *Intransitive Dice*

We start by computing the expected value and variance for each die. Let X_1, X_2, and X_3 be the random variables representing the outcome after rolling die 1, 2, and 3. Then

$$\begin{aligned}
\mathbb{E}[X_1] &= \frac{1}{3} \cdot (2+4+9) = 5 \\
\mathbb{E}[X_2] &= \frac{1}{3} \cdot (1+6+8) = 5 \\
\mathbb{E}[X_3] &= \frac{1}{3} \cdot (3+5+7) = 5 \\
\mathrm{Var}[X_1] &= \frac{1}{3} \cdot ((2-5)^2 + (4-5)^2 + (9-5)^2) = \frac{26}{3} \\
\mathrm{Var}[X_2] &= \frac{1}{3} \cdot ((1-5)^2 + (6-5)^2 + (8-5)^2) = \frac{26}{3} \\
\mathrm{Var}[X_3] &= \frac{1}{3} \cdot ((3-5)^2 + (5-5)^2 + (7-5)^2) = \frac{8}{3}\,.
\end{aligned}$$

Thus, the initial claims of both Alice and Bob are correct: the expected value is always the same, and the first two dice have a higher variance. However, they both draw incorrect conclusions from this. Both the expected value and the variance do not give any reasonable clues on which die to choose. This becomes most obvious when we replace "9" on the first die by "99999". This has a (huge) impact on the variance and expected value of X_1, but has no impact on who wins the game.

Instead, Clara is right. It is not too difficult to verify that

$$\Pr[X_3 > X_1] = \Pr[X_2 > X_3] = \Pr[X_1 > X_2] = \frac{5}{9}$$

(e.g. one could simply list the 9 possible outcomes for each pair of dice). Hence, no matter which die Bob chooses, Clara can always choose a die that gives her a $\frac{5}{9}$ chance of winning.