S9

Exercise S9.1 – *Boosting the success probability of Monte Carlo Algorithms*

Alice, Bob and Claire have learned that Monte Carlo algorithms can often be improved by running them multiple times. They are each given a blackbox [1] Monte Carlo algorithm $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ with a certain bound on the error. The task is to construct new algorithms $\mathcal{A}\_improved$, $\mathcal{B}\_improved$, and $\mathcal{C}\_improved$ resp. (using $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ resp. as subroutine) with a success probability of at least 99%. Try to keep the number of calls to the subroutines small.

(a) Given a graph (V, E), Alice tries to find a vertex set $\emptyset \neq S \subsetneq V$ that minimizes $|\delta(S)|$. She is given a Monte Carlo Algorithm $\mathcal{A}$ that, given a graph, returns some vertex set $\emptyset \neq S \subsetneq V$. With probability at least $p_{\mathcal{A}} \geq 1/\binom{n}{2}$ this set minimizes $|\delta(S)|$. Use this algorithm $\mathcal{A}$, to construct another algorithm $\mathcal{A}\_improved$ for the same problem, that has a success probability of at least 99%. *Hint:* $1 + x \leq e^x$ *for* $x \in \mathbb{R}$.

(b) Bob wants to check if a number is prime. He already knows a Monte Carlo algorithm $\mathcal{B}$ which takes as input a natural number N. If N is prime, the algorithm always returns 'prime'. If N is not prime, the algorithm returns 'not a prime' with probability $p_B \geq 1/2$. Use Bobs algorithm $\mathcal{B}$, to construct another algorithm $\mathcal{A}\_improved$ for the same problem, that has a success probability of at least 99%.

(c) Claire chose the most difficult problem. She wants to determine if a given graph has an Hamiltonian cycle. She thought of an algorithm $\mathcal{C}$ that, given a graph, outputs 'YES' or 'NO'. If a graph G has (does not have) a Hamiltonian cycle, the algorithms $\mathcal{C}(G)$ returns 'YES' ('No') with probability $p_{\mathcal{C}} \geq 3/4$. Help Claire to find an algorithm C_*improved* that is correct with probability at least 99%.

(d) Unfortunately, you forgot what Clair's initial algorithm $\mathcal{C}$ was. Thus, you have to find an initial algorithm yourself. Describe a fast Monte Carlo algorithm that, given a graph, outputs the correct answer 'YES' or 'NO' with probability 1/2. Can you boost this algorithm to a success probability of 99%?

(e) **Difficult**: Assume you are given the Monte Carlo algorithm $\mathcal{C}$ from part (c). Can you construct an algorithm that not only determines whether there is a Hamiltonian cycle, but even computes one if it exists. Of course, again only with a success probability of 99%.