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Algorithms and Probability **Session Sheet S12** **FS 2026**

For the exercise sessions on 21 May 2026.

Exercise S12.1 – *Integrality and Matching*

We have already seen Frobenius' Theorem: *Let $k \geq 1$. Every k-regular bipartite graph contains a perfect matching.* The proof we have seen uses Hall's theorem. In this exercise, we want to find another proof using our knowledge about flows.

Let $G = (A \dot\cup B, E)$ be a k-regular graph for $k \geq 1$.

(a) Describe how to model bipartite matching on G as a flow problem in a network $N = (V, A, c, s, t)$.

(b) Construct explicitly a (not necessarily integer!) flow f on N with $\text{val}(f) = n$. Conclude that $\text{maxflow}(N) = n$.
Hint: you don't have to construct an integral flow. Make sure to define the flow value on all edges of your network N!

(c) Using the result from (b), show that there exists an integer valued flow f' with $\text{val}(f') = n$. Using f', prove that G contains a perfect matching M.

Solution S12.1 – *Integrality und Matching*

(a) We do the same construction as always: we direct all edges from A to B. Furthermore, we add a source s, and all edges (s, a) for $a \in A$, and we add a sink t and all edges (b, t) for $b \in B$. Each edge gets capacity 1. There is a one to one correspondence between matchings in G and *integral* flows of value $|A| = |B|$ in N. (See Lemma 3.15.)

(b) For our flow f, we set $f(e) = 1/k$ for all edges between A and B. For the remaining edges, we set the flow to 1. Because all capacities are 1, they are satisfied. Flow consevation is also satisfied: each vertex in A has one incoming edge with flow value 1 and k outgoing edges with flow value $1/k$ each. For vertices in B it is exactly the other way around. Here we used that G is k-regular. The value of this flow equals $|A| = |B|$. The cut $cap(\{s\}, U \cup W \cup \{t\})$ has capacity $|A|$. Thus, f is a maximal flow.

(c) By Theorem 3.12, the maximum flow in N and the maximum integral flow in N have the same value. Thus, by (b), there is an integral flow of value $|A|$. By Lemma 3.15, G has a matching of size $|A| = |B|$. This is already a perfect matching.