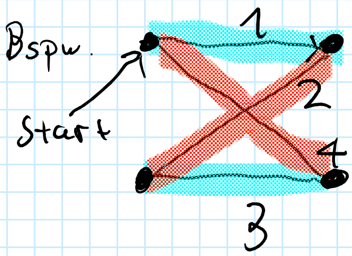
Notes S3

04 March 2026 00:29

Exercise S3.1 – *Bipartite Matching*

Let $G = (A \dot\cup B, E)$ be a bipartite graph.

(a) Prove or disprove: If G has a Hamiltonian cycle, then G contains two disjoint perfect matchings.

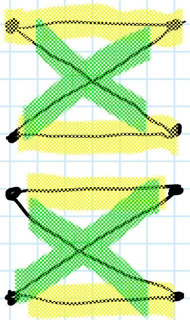


Bc G is bipartite a Ham. cycle C is of even length

$\Rightarrow$ We can always create 2 disjoint matchings by adding all odd numbered edges to one matching and all even numbered edges to the other (starting from any given vertex)

The matchings must cover all vertices since C is Hamiltonian.

(b) Prove or disprove: If G contains two disjoint perfect matchings, then G has a Hamiltonian cycle.

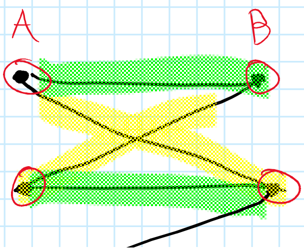


Counterexample:

2 perf. disjoint matchings, but no Ham. cycle, since G is not connected

(c) Let $A' \subseteq A$ and $B' \subseteq B$. Assume that there is a matching M_A covering A' and a matching M_B covering B' (these two matchings do not have to be disjoint!). Show that there is a matching M that covers both A' and B' (i.e. it covers $A' \cup B'$). (*Hint:* Which properties has the graph $(V, M_A \cup M_B)$? Try to build a matching using only edges in $M_A \cup M_B$.)

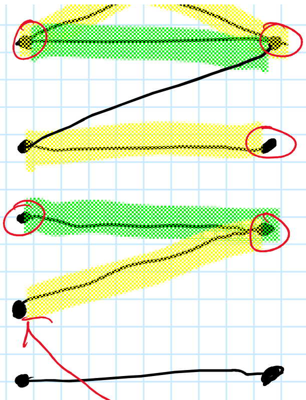
From lecture: $M_A \cup M_B$ consists of vertex disjoint cycles and paths.



Vertices that must be covered

Case cycle: Similar process to A, use every odd edge.

edge.



Case 2

Case 3

not this one bc it's in A and not covered by M_A

(c) As seen in the lecture, $M_A \cup M_B$ consists of vertex disjoint cycles and paths. We define a matching M as follows. 1 For each cycle we choose every second edge for our matching M (as in (a)). This ensures that all vertices of cycles are covered by M. 2 For paths of odd length (odd number of edges), we choose every second edge, starting with the first. This again ensures that all edges of such paths are covered by M (because we choose both the first and the last edge of the path). 3 It remains to consider paths of even length. Let P be such a path. We need two properties of P. First, the edges of P are alternatingly part of M_A and M_B. Since there is an even number of edges, we may assume without loss of generality that the first edge is in M_A and the last edge is in M_B. Second, the vertices of P are alternatingly in A and in B. Since there is an odd number of vertices, we may assume without loss of generality that both the start and endpoint of P are in A. Hence, the endpoint of P is in A but it is only covered by an edge in M_B. Thus, it cannot be part of $A' \cup B'$, meaning that we do not have to cover it. Hence, choosing every second edge of P starting with the first (i.e. choosing $P \cap M_A$) covers all *relevant* vertices of P.