Week 4

Colouring (with exercise)

Probability Intro/Recap

Combinatorics Recap

Infos

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Material:

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Colouring

Colouring

For every graph G, there exists an ordering of the vertices $V = \{v_1, \ldots, v_n\}$ such that the Greedy algorithm uses exactly $\chi(G)$ colors.

Intuition:

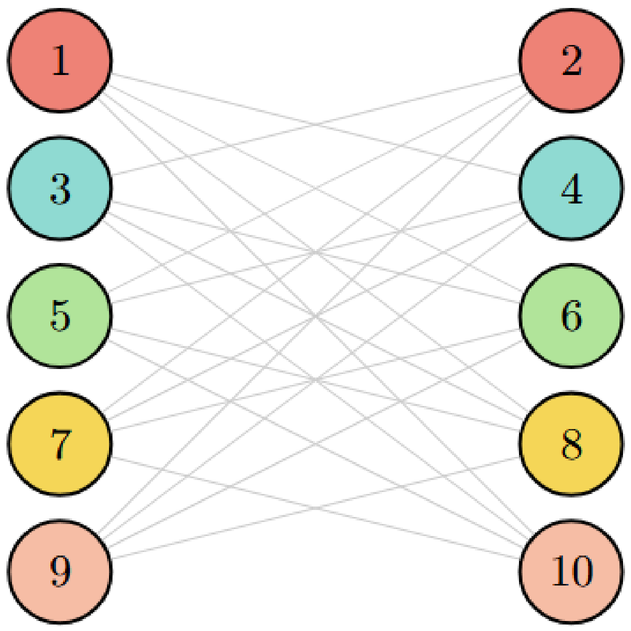
While the Greedy algorithm is sensitive to the sequence of vertices, an optimal coloring always "exists" within the algorithm's reach. One such perfect ordering is to group all vertices of the same color class together from an optimal coloring:

$$\underbrace{v_1, \ldots, v_i}_{\text{Color 1}} \to \underbrace{v_{i+1}, \ldots, v_j}_{\text{Color 2}} \to \cdots \to \underbrace{v_k, \ldots, v_n}_{\text{Color } \chi(G)}$$

Colouring

Worst-Case Greedy Coloring

Observation: There exist bipartite graphs and a vertex ordering $V = \{v_1, \ldots, v_n\}$ for which the Greedy algorithm requires $|V|/2$ colors.



Note: Although this graph is bipartite ($\chi(G) = 2$), the ordering $(1, 2, 3, \ldots, 10)$ forces the algorithm to use 5 different colors.

Colouring

GREEDY-FÄRBUNG (G)

1: wähle eine beliebige Reihenfolge der Knoten: $V = \{v_1, \ldots, v_n\}$
2: $c[v_1] \leftarrow 1$
3: **for** $i = 2$ **to** $i = n$ **do**
4: $\quad c[v_i] \leftarrow \min\{k \in \mathbb{N} \mid k \neq c(u)$ für alle $u \in N(v_i) \cap \{v_1, \ldots, v_{i-1}\}\}$

Beobachtung:
Gilt für die (gewählte) Reihenfolge $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leq k \quad \forall\, 2 \leq i \leq n$,
dann benötigt der Greedy-Algorithmus höchstens **k+1** viele Farben.

Extra exercise colouring (b)

(b) Let $G = (V, E)$ be a graph, and let $S \subset V$ be a set of all vertices with degree larger than $k-1$. Show that if $|S| \leq k$, there is a proper coloring of G into at most k colors. Describe an algorithm finding such a coloring in time $O(|V| + |E|)$.

(b) Consider an order $v_1, v_2, \dots v_n$ on vertices V, s.t. $\{v_1, \dots v_{|S|}\} = S$ — we first put all vertices of degree larger than $k-1$ and then remaining vertices. We will show that for every j we have $|N(v_j) \cap \{v_1, \dots v_{j-1}\}| \leq k-1$. Indeed, if $j \leq |S|$, then $|N(v_j) \cap \{v_1, \dots v_{j-1}\}| \leq j-1 \leq k-1$ since $|S| \leq k$. On the other hand, if $j > |S|$, then $v_j \notin S$ and therefore $|N(v_j) \cap \{v_1, \dots v_{j-1}\}| \leq |N(v_j)| \leq k-1$.

Now a greedy algorithm as in the solution of (a) will iterate over $j \in \{1, \dots n\}$ and color v_j into the first color not present among the colors of the $N(v_j) \cap \{v_1, \dots v_{j-1}\}$. Since we showed that this set is always of size at most $k-1$, there is always free color in $[k]$ to use.

Colouring

General idea behind exercise from last week:

Heuristik:

v_n := Knoten vom kleinsten Grad. Lösche v_n.
v_{n-1} := Knoten vom kleinsten Grad im Restgraph. Lösche v_{n-1}. Iteriere.

“Hebe dir die einfachen Knoten für den Schluss auf.”

gegensätzliche Idee:

- Starte mit einem einfachen Teilgraphen und färbe ihn zuerst.
- Verwende neue Farben für den Rest.

GREEDY-FÄRBUNG (G)

wähle eine beliebige Reihenfolge der Knoten: $V = \{v_1, \ldots, v_n\}$
$c[v_1] \leftarrow 1$
for $i = 2$ **to** n **do**
    $c[v_i] \leftarrow \min\{k \in \mathbb{N} \mid k \neq c(u)$ für alle $u \in N(v_i) \cap \{v_1, \ldots, v_{i-1}\}\}$

Colouring

Satz 1.65. Ist $G = (V, E)$ ein Graph und $k \in \mathbb{N}$ eine natürliche Zahl mit der Eigenschaft, dass jeder induzierte Subgraph von G einen Knoten mit Grad höchstens k enthält, so gilt $\chi(G) \leq k + 1$ und eine $(k + 1)$-Färbung lässt sich in Zeit $O(|E|)$ finden.

- In practice:

Heuristik:
v_n := Knoten vom kleinsten Grad. Lösche v_n.
v_{n-1} := Knoten vom kleinsten Grad im Restgraph. Lösche v_{n-1}.
Iteriere.

Falls G=(V,E) erfüllt:
In jedem Subgraphen gibt es einen Knoten mit Grad $\leq k$
⇒ Heuristik liefert Reihenfolge $v_1, \ldots, v_n$ für die der Greedy-Algorithmus höchstens k+1 Farben benötigt

Colouring

- This is also possible with heuristic:

The heuristic always finds a coloring with 2 colors for **trees**.

The heuristic finds a coloring with ≤ 6 colors for **planar graphs**.

If $G = (V, E)$ is connected and there exists a vertex $v \in V$ such that $\deg(v) < \Delta(G)$, then the heuristic provides an ordering for which the Greedy algorithm requires at most $\Delta(G)$ colors.

Exercise S5

2nd colouring exercise

(this time it’s an official session exercise)

Colouring

Satz 1.67. Jeden 3-färbbaren Graphen $G = (V, E)$ kann man in Zeit $O(|E|)$ mit $O(\sqrt{|V|})$ Farben färben.

Exercise S5.1 – *3-colorable graphs*

In the lecture you have seen an algorithm that finds an $O\left(\sqrt{|V|}\right)$-coloring for every 3-colorable graph (V, E). Fix $\alpha = \sqrt{|V|}$. The algorithm goes as follows.

1. While there is a vertex v with degree at least α, color v with a new color, color the neighbors of v with two additional new colors, and delete v and all its neighbors from the graph.

2. Color all remaining vertices with at most $\alpha + 1$ colors.

We want to analyze and generalize this algorithm.

(a) Where in the algorithm do we use that the graph is 3-colorable?

(b) How many colors do we need at most (depending on α)? Show that for $\alpha = \sqrt{|V|}$ we only need $O\left(\sqrt{|V|}\right)$ colors.

(c) Show that your bound in (a) is tight. More precisely, create a 3-colorable graph on which the algorithm uses $\Omega\left(\sqrt{|V|}\right)$ colors.

(d) Can you describe an algorithm that finds a $O\left(|V|^{\frac{2}{3}}\right)$-coloring for every 4 colorable graph?

(e) Can you describe an algorithm that finds a $O\left(|V|^{\frac{q-2}{q-1}}\right)$-coloring for every q colorable graph, where $q \geq 2$ is a constant?

Probability Intro/Recap

Probability

What do you use it for?

- Everything
- This course → algorithms
- Machine Learning
- Probability and Statistics in 2nd year

Probability

Definition 2.1. Ein *diskreter Wahrscheinlichkeitsraum* ist bestimmt durch eine *Ergebnismenge* $\Omega = \{\omega_1, \omega_2, \ldots\}$ von *Elementarereignissen*. Jedem Elementarereignis ω_i ist eine *(Elementar-)Wahrscheinlichkeit* $\Pr[\omega_i]$ zugeordnet, wobei wir fordern, dass $0 \leq \Pr[\omega_i] \leq 1$ und

$$\sum_{\omega \in \Omega} \Pr[\omega] = 1.$$

Eine Menge $E \subseteq \Omega$ heisst *Ereignis*. Die Wahrscheinlichkeit $\Pr[E]$ eines Ereignisses ist definiert durch

$$\Pr[E] := \sum_{\omega \in E} \Pr[\omega].$$

Ist E ein Ereignis, so bezeichnen wir mit $\bar{E} := \Omega \setminus E$ das *Komplementärereignis* zu E.

Probability

Lemma 2.2. For events A, B, the following holds:

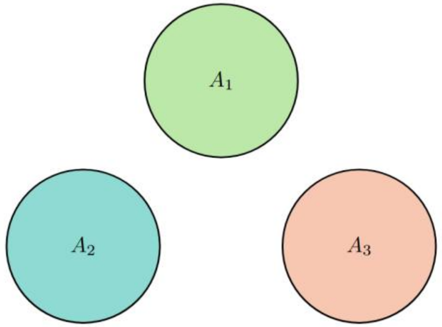
1. $\Pr[\emptyset] = 0, \Pr[\Omega] = 1$.
2. $0 \leq \Pr[A] \leq 1$.
3. $\Pr[\bar{A}] = 1 - \Pr[A]$.
4. If $A \subseteq B$, then it follows that $\Pr[A] \leq \Pr[B]$.

Probability

Addition Law

If the events $A_1, \ldots, A_n$ are **pairwise disjoint** (meaning for all pairs $i \neq j$, the intersection is empty: $A_i \cap A_j = \emptyset$), then it follows:

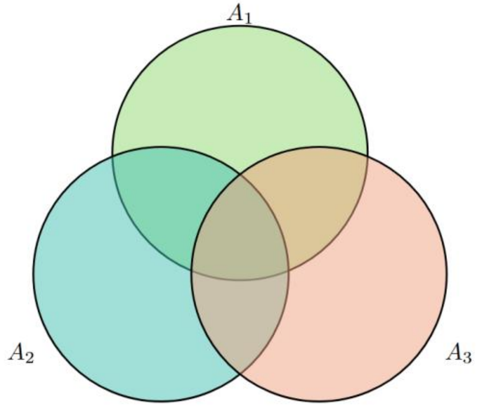
$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \Pr[A_i]$$



Union Bound

For **arbitrary** events $A_1, \ldots, A_n$, the following inequality holds:

$$\Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \Pr[A_i]$$

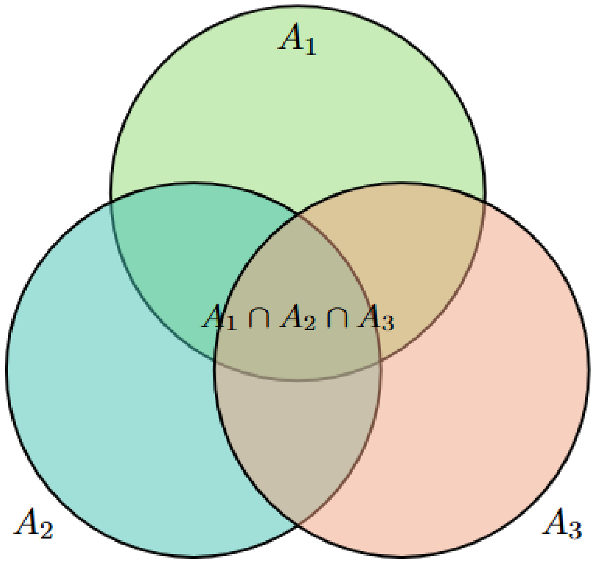


Probability

Inclusion-Exclusion Principle ($n = 3$)

The probability of the union of three events is calculated by:

1. Adding the areas of A_1, A_2, and A_3.
2. Subtracting the areas where two events overlap (they were counted twice).
3. Adding back the center where all three overlap (it was removed entirely in step 2).



$$\Pr\left[\bigcup_{i=1}^{3} A_i\right] = \sum \Pr[A_i] - \sum \Pr[A_i \cap A_j] + \Pr[A_1 \cap A_2 \cap A_3]$$

Probability

Satz 2.5. *(Siebformel, Prinzip der Inklusion/Exklusion)*
Für Ereignisse $A_1, \dots, A_n$ $(n \geq 2)$ gilt:

$$\begin{aligned}
\Pr\left[\bigcup_{i=1}^{n} A_i\right] &= \sum_{l=1}^{n} (-1)^{l+1} \sum_{1 \leq i_1 < \cdots < i_l \leq n} \Pr[A_{i_1} \cap \cdots \cap A_{i_l}] \\
&= \sum_{i=1}^{n} \Pr[A_i] - \sum_{1 \leq i_1 < i_2 \leq n} \Pr[A_{i_1} \cap A_{i_2}] \\
&\quad + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \Pr[A_{i_1} \cap A_{i_2} \cap A_{i_3}] - \cdots \\
&\quad + (-1)^{n+1} \cdot \Pr[A_1 \cap \cdots \cap A_n].
\end{aligned}$$

Probability

Definition 2.8. A und B seien Ereignisse mit $\Pr[B] > 0$. Die *bedingte Wahrscheinlichkeit* $\Pr[A|B]$ von A gegeben B ist definiert durch

$$\Pr[A|B] := \frac{\Pr[A \cap B]}{\Pr[B]}.$$

Satz 2.10. *(Multiplikationssatz)* Seien die Ereignisse $A_1, \ldots, A_n$ gegeben. Falls $\Pr[A_1 \cap \cdots \cap A_n] > 0$ ist, gilt

$$\Pr[A_1 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdots \Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

Umfrage

A&W G-13

Probability

Satz 2.13. *(Satz von der totalen Wahrscheinlichkeit)* Die Ereignisse $A_1, \dots, A_n$ seien paarweise disjunkt und es gelte $B \subseteq A_1 \cup \dots \cup A_n$. Dann folgt

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i].$$

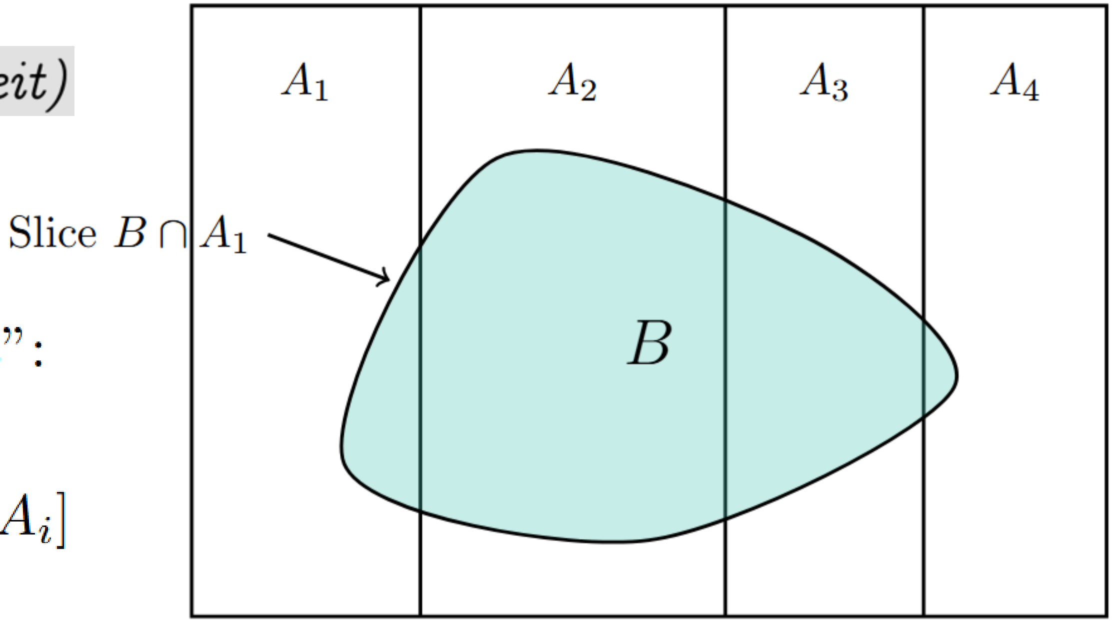
Analog gilt für paarweise disjunkte Ereignisse $A_1, A_2, \dots$ mit $B \subseteq \bigcup_{i=1}^{\infty} A_i$, dass

$$\Pr[B] = \sum_{i=1}^{\infty} \Pr[B|A_i] \cdot \Pr[A_i].$$

Probability

Given that $A_1, \ldots, A_n$ are **pairwise disjoint** and $B \subseteq \bigcup A_i$:

Satz 2.13. *(Satz von der totalen Wahrscheinlichkeit)*



The total probability is the sum of these "slices":

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B \cap A_i]$$

Since $\Pr[B \cap A_i] = \Pr[B|A_i] \cdot \Pr[A_i]$, we get:

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i]$$

Probability

Satz 2.15. *(Satz von Bayes)* Die Ereignisse $A_1, \ldots, A_n$ seien paarweise disjunkt. Ferner sei $B \subseteq A_1 \cup \cdots \cup A_n$ ein Ereignis mit $\Pr[B] > 0$. Dann gilt für ein beliebiges $i = 1, \ldots, n$

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{n} \Pr[B|A_j] \cdot \Pr[A_j]}.$$

Analog gilt für paarweise disjunkte Ereignisse $A_1, A_2, \ldots$ mit $B \subseteq \bigcup_{i=1}^{\infty} A_i$, dass

$$\Pr[A_i|B] = \frac{\Pr[A_i \cap B]}{\Pr[B]} = \frac{\Pr[B|A_i] \cdot \Pr[A_i]}{\sum_{j=1}^{\infty} \Pr[B|A_j] \cdot \Pr[A_j]}.$$

Probability

Scenario: A disease affects 1% of the population. A test is 99% accurate. **If a test is positive, what is the probability of having the disease?**

Example table with population of 10'000:

Status	**Test Positive (+)**	**Test Negative (-)**	**Total**
Sick (D)	99	1	100
Healthy ($\bar{D}$)	99	9,801	9,900
Total	**198**	9,802	10,000

Probability

Step 1: Identify the Given Probabilities

- **Prior Probability** (Sick): $\Pr[D]=0.01$
- **Prior Probability** (Healthy): $\Pr[\bar{D}]=0.99$
- **Sensitivity** (True Positive): $\Pr[+|D]=0.99$
- **False Positive Rate**: $\Pr[+|\bar{D}]=0.01$

Probability

Want to find:

$$\Pr[D|+] = \frac{\Pr[D] \cdot \Pr[+|D]}{\Pr[+]}$$

Probability

Step 2: Calculate the Total Probability of Testing Positive $\Pr[+]$

Using the **Satz von der totalen Wahrscheinlichkeit**, we sum the two ways to get a positive result:

$$\Pr[+] = (\text{Sick and Test Pos}) + (\text{Healthy and Test Pos})$$
$$\Pr[+] = (\Pr[D] \cdot \Pr[+|D]) + (\Pr[\bar{D}] \cdot \Pr[+|\bar{D}])$$
$$\Pr[+] = (0.01 \cdot 0.99) + (0.99 \cdot 0.01)$$
$$\Pr[+] = 0.0099 + 0.0099 = \mathbf{0.0198}$$

Step 3: Apply Bayes' Theorem

We find the proportion of "True Positives" out of "All Positives":

$$\Pr[D|+] = \frac{\Pr[D] \cdot \Pr[+|D]}{\Pr[+]}$$

$$\Pr[D|+] = \frac{0.0099}{0.0198}$$

$$\Pr[D|+] = \mathbf{0.5}\ (50\%)$$

Probability

Definition 2.18. Die Ereignisse A und B heissen *unabhängig*, wenn gilt

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B].$$

Probability

Given $\Pr[B] \neq 0$, Intuition: A is not dependent on B

$$\Pr[A] = \Pr[A|B]$$

$$\downarrow$$

$$\Pr[A] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

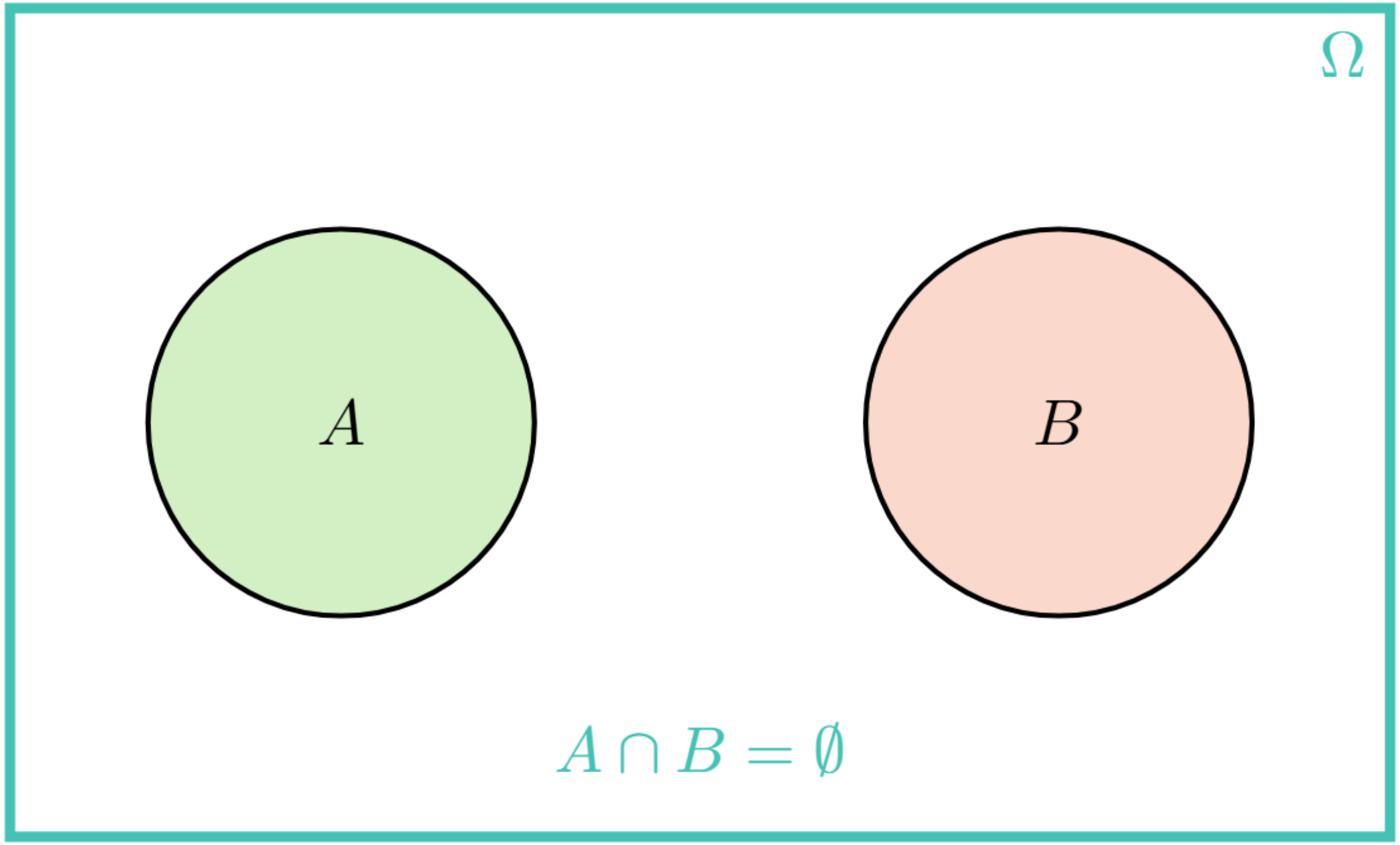
$$\downarrow$$

$$\Pr[\mathbf{A} \cap \mathbf{B}] = \Pr[\mathbf{A}] \cdot \Pr[\mathbf{B}]$$

Probability

Are A and B independent?

Assume $\Pr[A] > 0$ and $\Pr[B] > 0$



$A \cap B = \emptyset \implies \Pr[A \cap B] = 0$

$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = 0$

NOOOOOO $\Pr[A \cap B] \neq \Pr[A] \cdot \Pr[B]$ $\Pr[A] \neq \Pr[A|B]$

Probability

Definition 2.22. Die Ereignisse $A_1, \ldots, A_n$ heissen *unabhängig*, wenn für alle Teilmengen $I \subseteq \{1, \ldots, n\}$ mit $I = \{i_1, \ldots, i_k\}$ gilt, dass

$$\Pr[A_{i_1} \cap \cdots \cap A_{i_k}] = \Pr[A_{i_1}] \cdots \Pr[A_{i_k}]. \tag{2.2}$$

Eine unendliche Familie von Ereignissen A_i mit $i \in \mathbb{N}$ heisst unabhängig, wenn (2.2) für jede endliche Teilmenge $I \subseteq \mathbb{N}$ erfüllt ist.

Lemma 2.23. Die Ereignisse $A_1, \ldots, A_n$ sind genau dann unabhängig, wenn für alle $(s_1, \ldots, s_n) \in \{0, 1\}^n$ gilt, dass

$$\Pr[A_1^{s_1} \cap \cdots \cap A_n^{s_n}] = \Pr[A_1^{s_1}] \cdots \Pr[A_n^{s_n}], \tag{2.3}$$

wobei $A_i^0 = \bar{A}_i$ und $A_i^1 = A_i$.

Probability

Lemma 2.24. Seien A, B und C unabhängige Ereignisse. Dann sind auch $A \cap B$ und C bzw. $A \cup B$ und C unabhängig.

Combinatorics Recap

Material: Dr. Geoffrey Ostrin

(my high school maths teacher ☺)

(crème de la crème of all maths teachers)

Combinatorics Recap

		replacement	
		Yes	No
order	Yes	n^k	$\frac{n!}{(n-k)!}$
	No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

n^k

Friday, 13 January 2023 08:17

Given n objects we select k s.t.

- order is important
- we have replacement

} no restriction

Combinatorics Recap

$$\underbrace{\underset{\_\_}{n}\quad \underset{\_\_}{n}\quad \underset{\_\_}{n}\quad \cdots \quad \underset{\_\_}{n}}_{k\text{-many}}$$

Product Principle

$$\underbrace{n \cdot n \cdot n \cdot \cdots \cdot n}_{k\text{-many}} = n^k.$$

Combinatorics Recap

order \ replacement	Yes	No
Yes	n^k	$\frac{n!}{(n-k)!}$
No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

Permutations

Friday, 13 January 2023 09:02

Given n objects we select k s.t.

- order is important
- we have NO replacement

} permutations

$$\underbrace{\underbrace{n}\ \underbrace{n-1}\ \underbrace{n-2}\ \dots\ \underbrace{n-(k-1)}}_{k\text{-many}}$$

/

Combinatorics Recap

Given n objects we select k s.t.

- order is important
- we have NO replacement

} permutations

$$\underbrace{\underline{n}\quad \underline{n-1}\quad \underline{n-2}\quad \ldots\quad \underline{n-(k-1)}}_{k\text{-many}}$$

P.P.

$$\frac{n\cdot(n-1)\cdot(n-2)\cdot\ \ldots\ \cdot(n-(k-1))\cdot(n-k)\cdot\ \ldots\ 3\cdot 2\cdot 1}{(n-k)\cdot(n-k-1)\cdot\ \ldots\ \cdot 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}$$

Combinatorics Recap

	replacement: Yes	replacement: No
order: Yes	n^k	$\frac{n!}{(n-k)!}$
order: No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

Combinations

Wednesday, 25 January 2023 13:22

Given n objects we select k s.t.

- order is NOT important
- we have NO replacement

} combinations

Example: How many combinations are there selecting 3 objects from the 5 $\{A, B, C, D, E\}$?

Combinatorics Recap

We start with the permutations ?

$$5 \cdot 4 \cdot 3 = 60$$

$$\frac{n!}{(n-k)!}$$

ABC ABD ABE ACD ACE ADE BCD BCE BDE CDE

ACB

BAC

BCA

CAB

CBA . . . .

60

. . . EDC

$6 = 3!$

$k!$

Combinatorics Recap

ABC ABD ABE ACD ACE ADE BCD BCE BDE CDE

ACB

BAC

BCA

CAB

CBA . . . . EDC

60

$6 = 3!$

$k!$

Since order now is not important

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)! \cdot k!} =: \binom{n}{k}$$

n choose k

Combinatorics Recap

Since order now is not important

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)! \cdot k!} =: \binom{n}{k}$$

n choose k

$$\binom{10}{3} := \frac{10!}{7! \cdot 3!} = \frac{10 \cdot \overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 120$$

Combinatorics Recap

order \ replacement	Yes	No
Yes	n^k	$\frac{n!}{(n-k)!}$
No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

Dots & Dividers

Wednesday, 1 February 2023 13:47

Given n objects we select k s.t.

- order is NOT important
- we have replacement

Combinatorics Recap

Example: From 5 objects, select 3 where order is not important and we have replacement.

$\{A, B, C, D, E\}$

AAA ✓

AAB ≡ ABA ≡ BAA ✓

ABC ≡ ACB ≡ BAC ≡ BCA ≡ CAB ≡ CBA

Combinatorics Recap

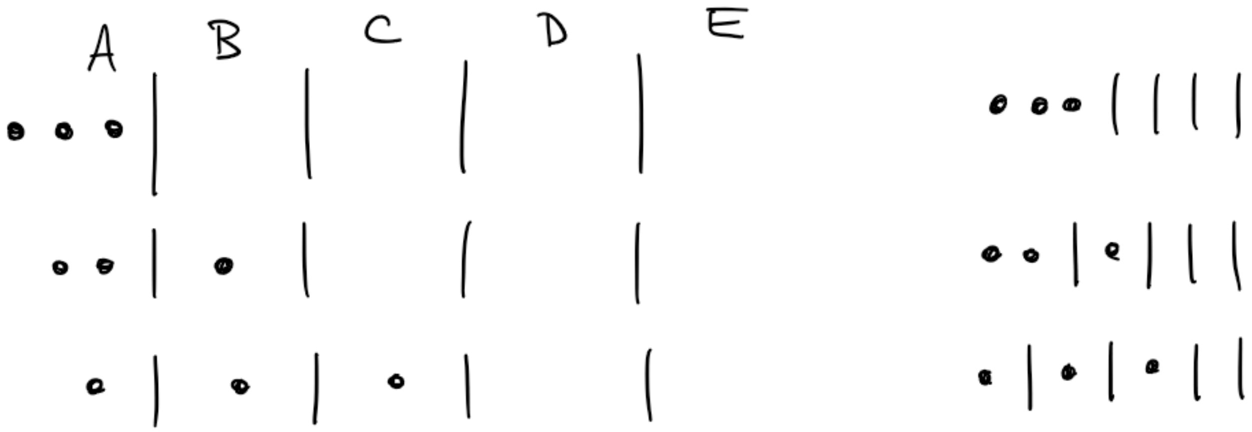
$\{A, B, C, D, E\}$

AAA ✓

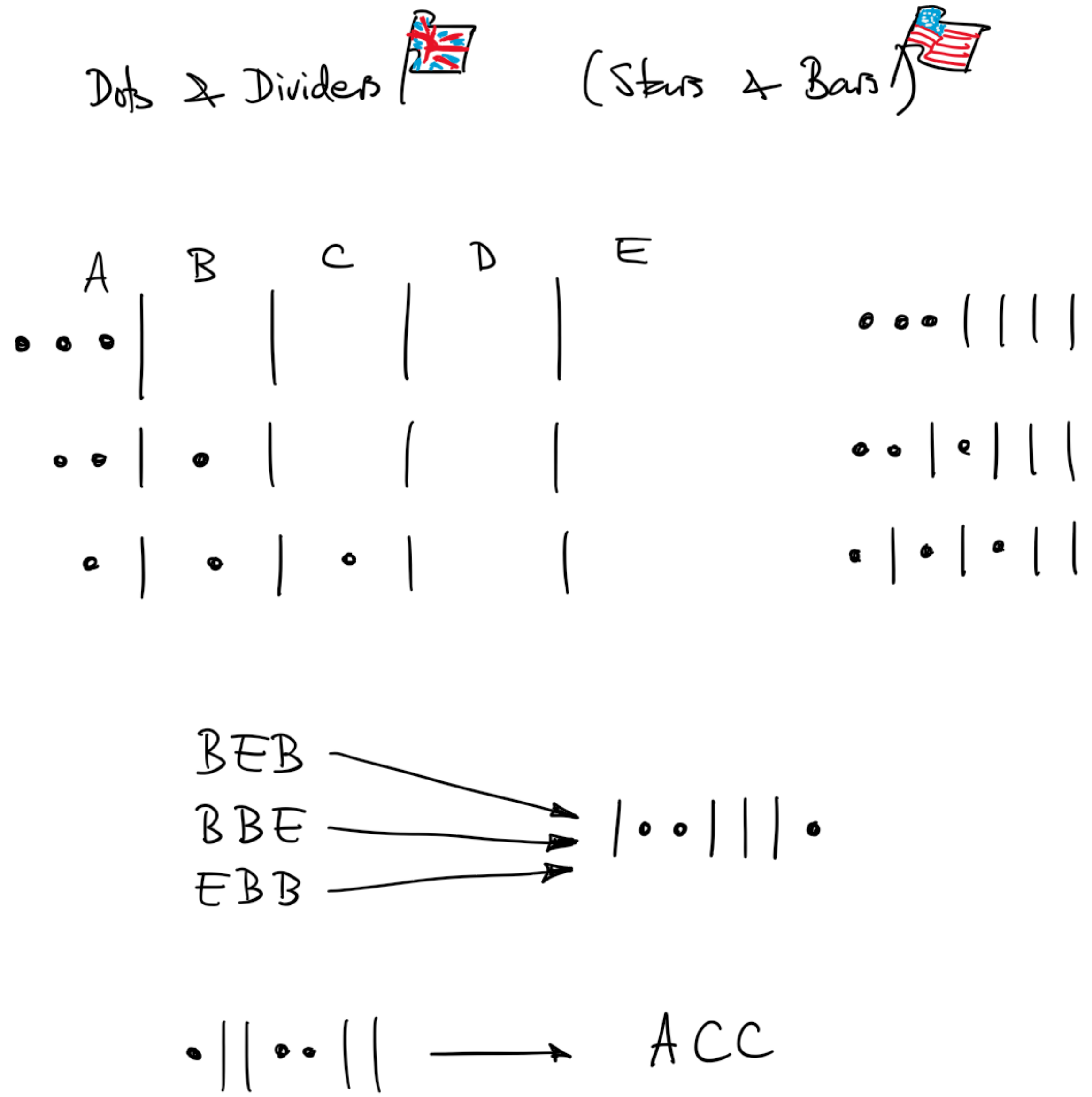
AAB ≡ ABA ≡ BAA ✓

ABC ≡ ACB ≡ BAC ≡ BCA ≡ CAB ≡ CBA

Dots & Dividers  (Stars & Bars) 



Combinatorics Recap



Combinatorics Recap

Hence, the problem reduces down to:

Ihmwc we mix up such a dot & divider picture?

↓

anagrams of • | | • • | |

_ •_ _ _ •_ •_ _   * | | * | | *

$$\binom{7}{3} = \frac{7!}{3!\cdot 4!} = \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1} = 35.$$

GENERALISE

n → n − 1 dividers

k → k dots

$$\binom{k+n-1}{k}$$

Combinatorics Recap

		replacement	
		Yes	No
order	Yes	n^k	$\frac{n!}{(n-k)!}$
	No	$\binom{n-1+k}{k}$	$\binom{n}{k}$