Week 5

Minitest 3

Probability Exercise

Combinatorics Recap

Random Variables

Distributions

Infos

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Material:

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Umfrage

A&W G-13

Probability Intro/Recap

Probability

Definition 2.18. Die Ereignisse A und B heissen *unabhängig*, wenn gilt

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B].$$

Probability

Given $\Pr[B] \neq 0$, Intuition: A is not dependent on B

$$\Pr[A] = \Pr[A|B]$$

$$\downarrow$$

$$\Pr[A] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

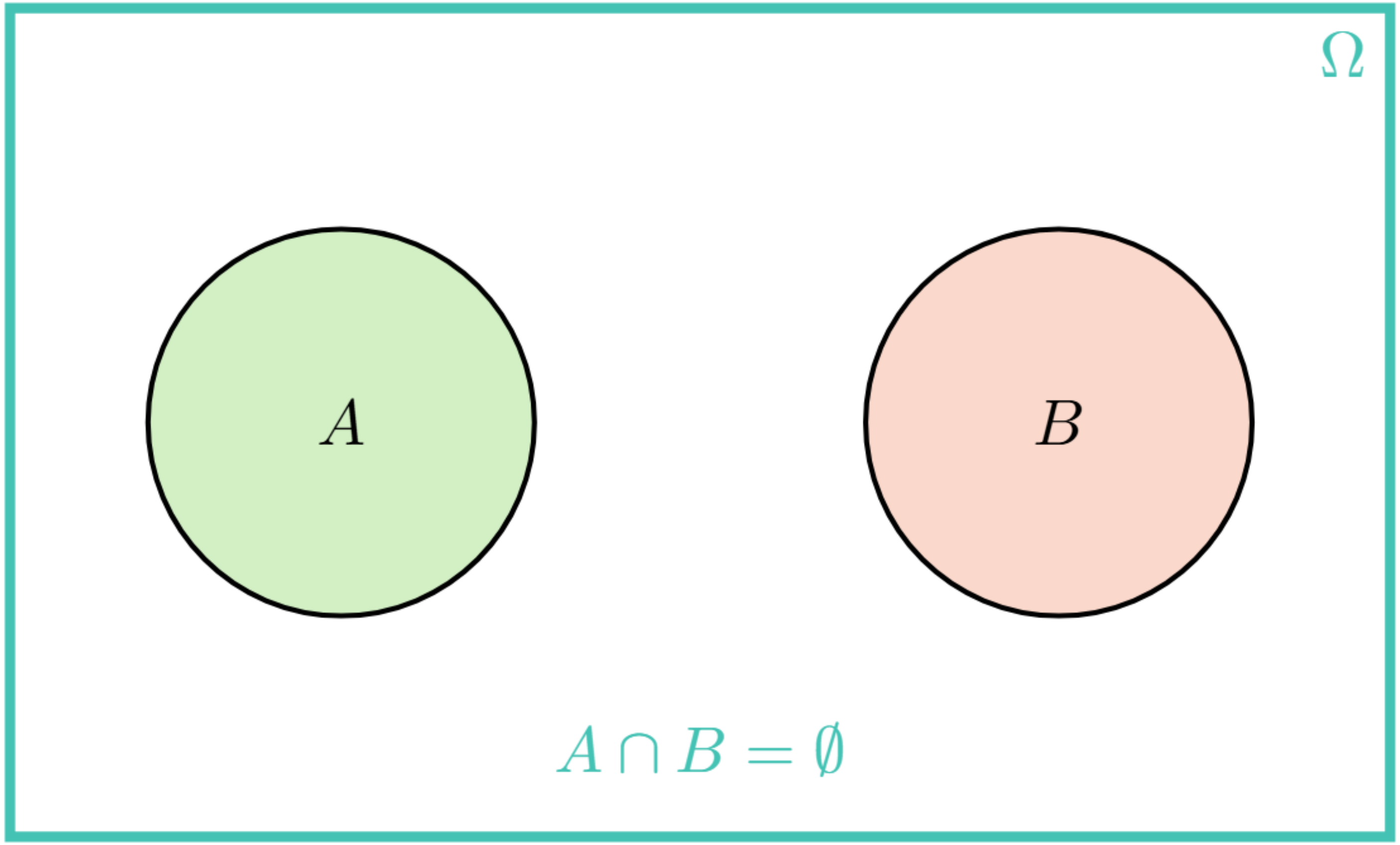
$$\downarrow$$

$$\Pr[\mathbf{A} \cap \mathbf{B}] = \Pr[\mathbf{A}] \cdot \Pr[\mathbf{B}]$$

Probability

Are A and B independent?

Assume $\Pr[A] > 0$ and $\Pr[B] > 0$



$$A \cap B = \emptyset \implies \Pr[A \cap B] = 0$$

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = 0$$

NOOOOOO $\Pr[A \cap B] \neq \Pr[A] \cdot \Pr[B]$ $\Pr[A] \neq \Pr[A|B]$

Probability

Definition 2.22. Die Ereignisse $A_1, \ldots, A_n$ heissen *unabhängig*, wenn für alle Teilmengen $I \subseteq \{1, \ldots, n\}$ mit $I = \{i_1, \ldots, i_k\}$ gilt, dass

$$\Pr[A_{i_1} \cap \cdots \cap A_{i_k}] = \Pr[A_{i_1}] \cdots \Pr[A_{i_k}]. \tag{2.2}$$

Eine unendliche Familie von Ereignissen A_i mit $i \in \mathbb{N}$ heisst unabhängig, wenn (2.2) für jede endliche Teilmenge $I \subseteq \mathbb{N}$ erfüllt ist.

Lemma 2.23. Die Ereignisse $A_1, \ldots, A_n$ sind genau dann unabhängig, wenn für alle $(s_1, \ldots, s_n) \in \{0, 1\}^n$ gilt, dass

$$\Pr[A_1^{s_1} \cap \cdots \cap A_n^{s_n}] = \Pr[A_1^{s_1}] \cdots \Pr[A_n^{s_n}], \tag{2.3}$$

wobei $A_i^0 = \bar{A}_i$ und $A_i^1 = A_i$.

Probability

Lemma 2.24. Seien A, B und C unabhängige Ereignisse. Dann sind auch $A \cap B$ und C bzw. $A \cup B$ und C unabhängig.

Exercise S6

Probability Space

Exercise S6.1 – *Probability Space*

(a) For each of the following subtasks, either define a probability space and events A and B (and C) with the described properties, or prove that such a space cannot exist. Make sure that you define both, the sample space ("Ergebnismenge") Ω and the probabilities of the atomic events ("Elementarereignisse").

   (i) $\Pr[A] = \frac{1}{4}$, $\Pr[B] = \frac{1}{3}$ and $\Pr[A \cup B] = \Pr[A] + \Pr[B]$.

   (ii) $\Pr[A] = \frac{1}{4}$, $\Pr[B] = \frac{1}{3}$ and $\Pr[A \cup B] < \Pr[A] + \Pr[B]$.

   (iii) $\Pr[A] = \Pr[B]$, $\Pr[A \cap B] = \frac{1}{4}$, and $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$ (that is A and B are independent).

   (iv) $\Pr[A] = \Pr[B] = \Pr[C] = \frac{5}{6}$ and $\Pr[A \cap B \cap C] = 0$.

(b) Samantha has a fair, six-sided die and a 5 CHF coin. She rolls the die and tosses the coin. Samantha considers her experiment a success if the coin shows a strictly larger value than the die (for the coin, heads is counted as 0; tails is counted as 5). Model her experiment with a suitable probability space. Explicitly define the event A that the experiment is a success and determine its probability $\Pr[A]$.

(c) Oliver owns three pairs of shoes – two blue pairs, and one yellow, which he stores unordered in his wardrobe. One morning, during a power outage, he has to put on his shoes in complete darkness. He randomly (uniformly at random) grabs two shoes from the wardrobe and tries to put them on.

We let A denote the event that he picked one left shoe and one right shoe (i.e. he is able to put on the shoes he picked), and we let B be the event that the two shoes he picked have the same color.

Model this setting as a probability space and compute $\Pr[A]$ and $\Pr[A|B]$.

Combinatorics Recap

Material: Dr. Geoffrey Ostrin

(my high school maths teacher ☺)

(crème de la crème of all maths teachers)

Combinatorics Recap

order \ replacement	Yes	No
Yes	n^k	$\frac{n!}{(n-k)!}$
No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

n^k

Friday, 13 January 2023 08:17

Given n objects we select k s.t.

- order is important
- we have replacement

} no restriction

Combinatorics Recap

$$\underbrace{\underset{\_\_}{n}\quad \underset{\_\_}{n}\quad \underset{\_\_}{n}\quad \cdots \quad \underset{\_\_}{n}}_{k\text{-many}}$$

Product Principle

$$\underbrace{n \cdot n \cdot n \cdot \cdots \cdot n}_{k\text{-many}} = n^k.$$

Combinatorics Recap

		replacement	
		Yes	No
order	Yes	n^k	$\frac{n!}{(n-k)!}$
	No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

Permutations

Friday, 13 January 2023 09:02

Given n objects we select k s.t.

- order is important
- we have NO replacement

} permutations

n $\quad n-1 \quad n-2 \quad \ldots \quad n-(k-1)$

k-many

/

Combinatorics Recap

Given n objects we select k s.t.

- order is important
- we have NO replacement

} permutations

$\underline{n}\ \underline{n-1}\ \underline{n-2}\ \dots\ \underline{n-(k-1)}$

$\underbrace{\qquad\qquad\qquad\qquad}_{k\text{-many}}$

P.P.

$$\frac{n\cdot(n-1)\cdot(n-2)\cdot\ \dots\ \cdot(n-(k-1))\cdot(n-k)\cdot\ \dots\ 3\cdot 2\cdot 1}{(n-k)\cdot(n-k-1)\cdot\ \dots\ \cdot 3\cdot 2\cdot 1} = \frac{n!}{(n-k)!}$$

Combinatorics Recap

order \ replacement	Yes	No
Yes	n^k	$\frac{n!}{(n-k)!}$
No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

Combinations

Wednesday, 25 January 2023 13:22

Given n objects we select k s.t.

- order is NOT important
- we have NO replacement

} combinations

Example: How many combinations are there selecting 3 objects from the 5 $\{A, B, C, D, E\}$?

Combinatorics Recap

We start with the permutations?

$$5 \cdot 4 \cdot 3 = 60$$

$$\frac{n!}{(n-k)!}$$

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB									⋮
BAC									
BCA									
CAB									
CBA	....							...	EDC

60

$6 = 3!$

$k!$

Combinatorics Recap

ABC ABD ABE ACD ACE ADE BCD BCE BDE CDE

ACB

BAC

BCA

CAB

CBA . . . . EDC

60

$6 = 3!$

$k!$

Since order now is not important

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)! \cdot k!} =: \binom{n}{k}$$

n choose k

Combinatorics Recap

Since order now is not important

$$\frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)! \cdot k!} =: \binom{n}{k}$$

n choose k

$$\binom{10}{3} := \frac{10!}{7! \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

Combinatorics Recap

		replacement: Yes	replacement: No
order	Yes	n^k	$\frac{n!}{(n-k)!}$
	No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Combinatorics Recap

Dots & Dividers

Wednesday, 1 February 2023 13:47

Given n objects we select k s.t.

- order is NOT important
- we have replacement

Combinatorics Recap

Example: From 5 objects, select 3 where order is not important and we have replacement.

{A, B, C, D, E}

AAA ✓

AAB ≡ ABA ≡ BAA ✓

ABC ≡ ACB ≡ BAC ≡ BCA ≡ CAB ≡ CBA

Combinatorics Recap

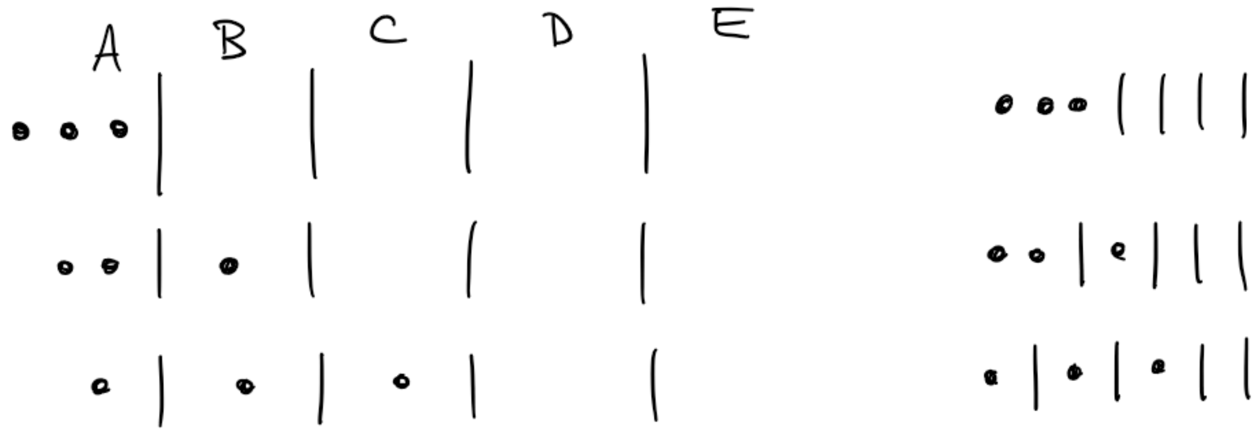
{A, B, C, D, E}

AAA ✓

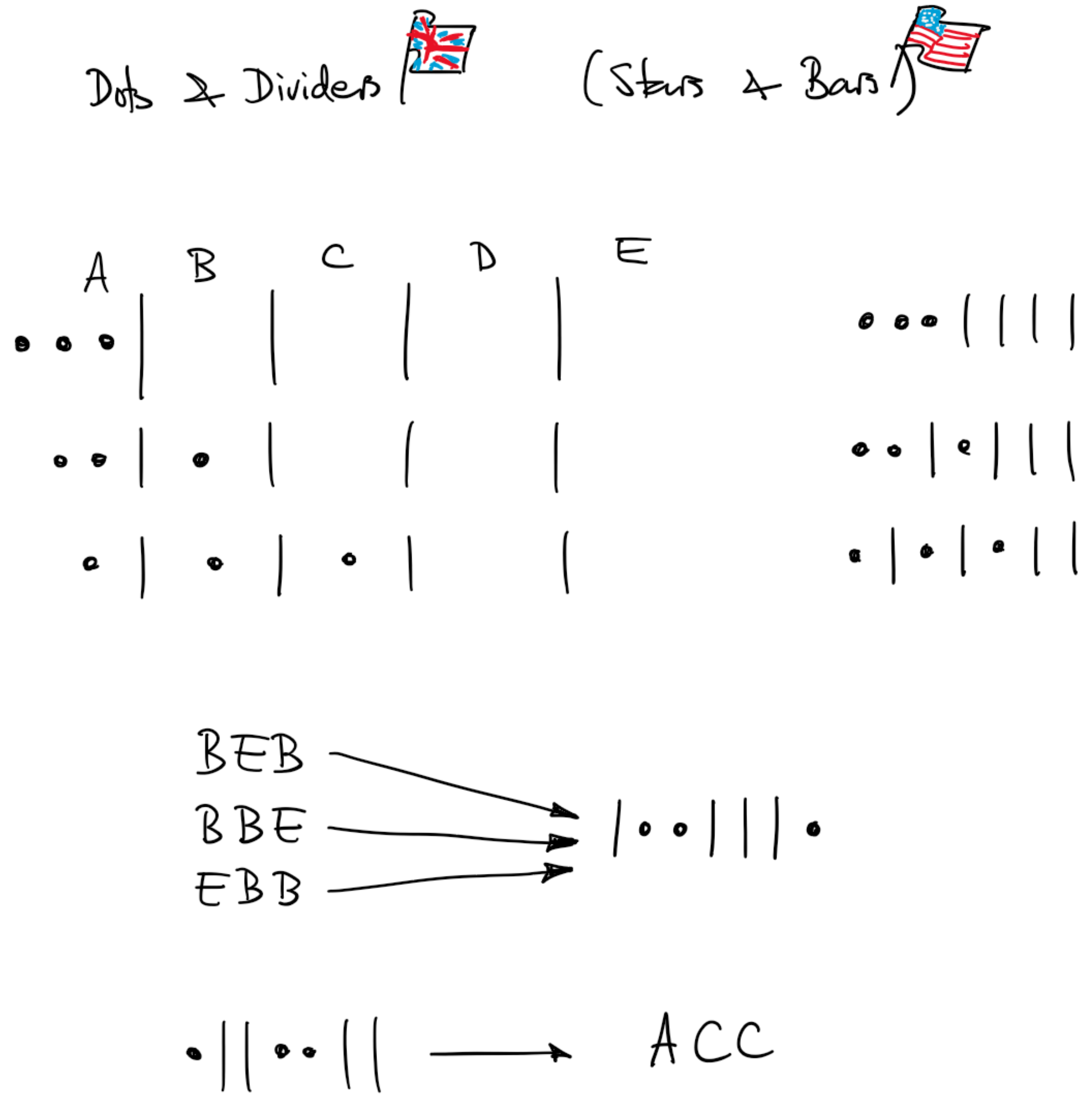
AAB ≡ ABA ≡ BAA ✓

ABC ≡ ACB ≡ BAC ≡ BCA ≡ CAB ≡ CBA

Dots & Dividers  (Stars & Bars) 



Combinatorics Recap



Combinatorics Recap

Hence, the problem reduces down to:

Ihmwc we mix up such a dot & divider picture?

anagrams of • || • • ||

* || * || *

$$\binom{7}{3} = \frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35.$$

GENERALISE

n → n−1 dividers

k → k dots

$$\binom{k+n-1}{k}$$

Combinatorics Recap

order \ replacement	Yes	No
Yes	n^k	$\frac{n!}{(n-k)!}$
No	$\binom{n-1+k}{k}$	$\binom{n}{k}$

Probability

Random Variables

Random Variables

Definition 2.25. Eine *Zufallsvariable* ist ein Abbildung $X \colon \Omega \to \mathbb{R}$, wobei Ω die Ergebnismenge eines Wahrscheinlichkeitsraumes ist.

Bei diskreten Wahrscheinlichkeitsräumen ist der *Wertebereich* einer Zufallsvariablen

$$W_X := X(\Omega) = \{x \in \mathbb{R} \mid \exists\, \omega \in \Omega \text{ mit } X(\omega) = x\}$$

Random Variables

Every (discrete) random variable X can be naturally assigned two real-valued functions.

Dichte(funktion)
The function $f_X : \mathbb{R} \to [0,1]$ is defined as:

$$f_X(x) = \Pr[X = x]$$

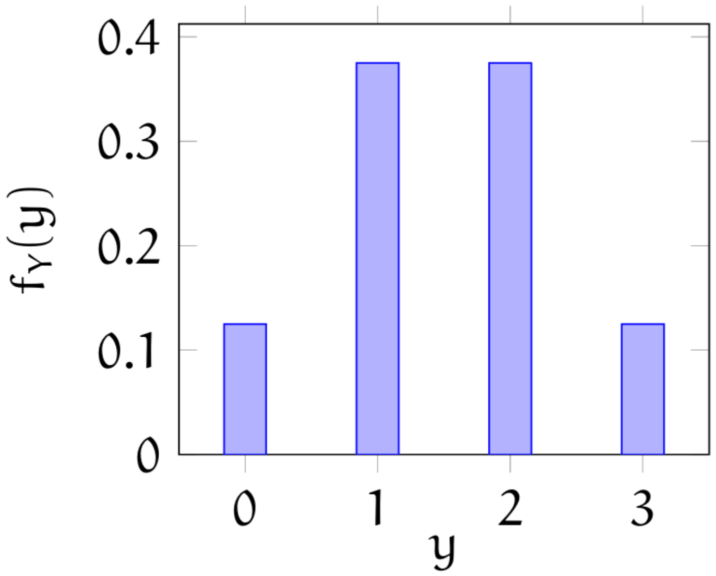
Verteilung(sfunktion)
The function $F_X : \mathbb{R} \to [0,1]$ is defined as:

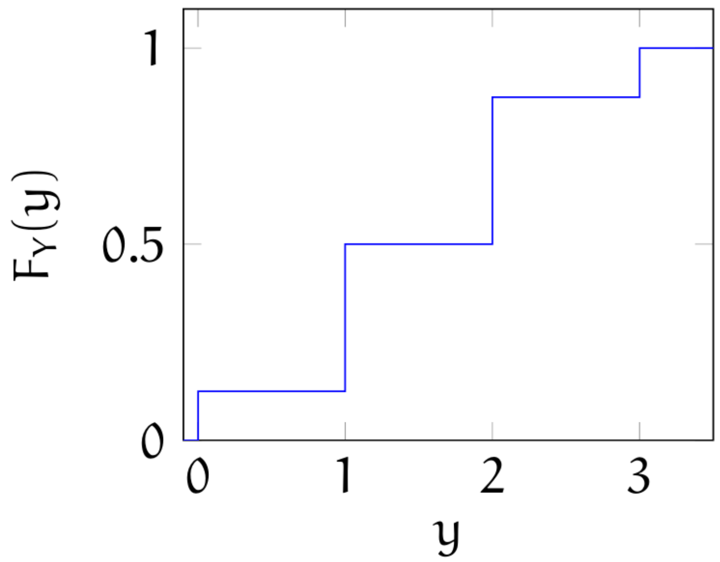
$$F_X(x) = \Pr[X \le x] = \sum_{x' \in W_X : x' \le x} \Pr[X = x']$$

Random Variables

Beispiel 2.26 *(Fortsetzung)* Für die Zufallsvariable Y erhalten wir

$$
\begin{aligned}
\Pr[Y=0] &= \Pr[ZZZ] = \frac{1}{8},\\
\Pr[Y=1] &= \Pr[KZZ] + \Pr[ZKZ] + \Pr[ZZK] = \frac{3}{8},\\
\Pr[Y=2] &= \Pr[KKZ] + \Pr[KZK] + \Pr[ZKK] = \frac{3}{8},\\
\Pr[Y=3] &= \Pr[KKK] = \frac{1}{8}.
\end{aligned}
$$





Expected Value and Variance

Random Variables

Definition 2.27. Zu einer Zufallsvariablen X definieren wir den *Erwartungswert* $\mathbb{E}[X]$ durch

$$\mathbb{E}[X] := \sum_{x \in W_X} x \cdot \Pr[X = x],$$

sofern die Summe absolut konvergiert. Ansonsten sagen wir, dass der Erwartungswert undefiniert ist.

(only holds for **discrete** random variables)

Random Variables

Definition 2.27. Zu einer Zufallsvariablen X definieren wir den *Erwartungswert* $\mathbb{E}[X]$ durch

$$\mathbb{E}[X] := \sum_{x \in W_X} x \cdot \Pr[X = x],$$

sofern die Summe absolut konvergiert. Ansonsten sagen wir, dass der Erwartungswert undefiniert ist.

Lemma 2.29. Ist X eine Zufallsvariable, so gilt:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega].$$

Random Variables

Satz 2.30. Sei X eine Zufallsvariable mit $W_X \subseteq \mathbb{N}_0$. Dann gilt

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i].$$

Random Variables

1. The Standard Definition

By definition, the expected value is the weighted sum of all possible outcomes:

$$E[X] = \sum_{x=1}^{\infty} x \cdot \Pr[X = x] = 1 \cdot \Pr[X = 1] + 2 \cdot \Pr[X = 2] + 3 \cdot \Pr[X = 3] + \dots$$

2. Expanding the Tail Probabilities

The tail probability $\Pr[X \geq k]$ is the sum of all individual probabilities from k onwards. We can write these out row by row:

$$\begin{aligned}
\Pr[X \geq 1] &= \Pr[X = 1] + \Pr[X = 2] + \Pr[X = 3] + \Pr[X = 4] + \dots \\
\Pr[X \geq 2] &= \qquad\qquad\quad\; \Pr[X = 2] + \Pr[X = 3] + \Pr[X = 4] + \dots \\
\Pr[X \geq 3] &= \qquad\qquad\qquad\qquad\qquad\quad\; \Pr[X = 3] + \Pr[X = 4] + \dots \\
\Pr[X \geq 4] &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \Pr[X = 4] + \dots \\
&=
\end{aligned}$$

Random Variables

3. Vertical Summation

If we sum all these rows **vertically**, we count the occurrences of each $\Pr[X = x]$ term:

- $\Pr[X = 1]$ appears **1** time (only in the first row).
- $\Pr[X = 2]$ appears **2** times (in the first and second row).
- $\Pr[X = 3]$ appears **3** times (in the first, second, and third row).

Summing the columns gives us:

$$\sum_{k=1}^{\infty} \Pr[X \geq k] = 1 \cdot \Pr[X = 1] + 2 \cdot \Pr[X = 2] + 3 \cdot \Pr[X = 3] + \ldots = E[X]$$

Random Variables

Satz 2.32. Sei X eine Zufallsvariable. Für paarweise disjunkte Ereignisse $A_1, \ldots, A_n$ mit $A_1 \cup \cdots \cup A_n = \Omega$ und $\Pr[A_1], \ldots, \Pr[A_n] > 0$ gilt

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i] \cdot \Pr[A_i].$$

Für paarweise disjunkte Ereignisse $A_1, A_2, \ldots$ mit $\bigcup_{i=1}^{\infty} A_k = \Omega$ und $\Pr[A_1], \Pr[A_2], \ldots > 0$ gilt analog

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{E}[X|A_i] \cdot \Pr[A_i].$$

Random Variables

Satz 2.33. *(Linearität des Erwartungswerts)* Für Zufallsvariablen $X_1, \ldots, X_n$ und $X := a_1X_1 + \ldots + a_nX_n + b$ mit $a_1, \ldots, a_n, b \in \mathbb{R}$ gilt

$$\mathbb{E}[X] = a_1\mathbb{E}[X_1] + \ldots + a_n\mathbb{E}[X_n] + b.$$

Random Variables

Beobachtung 2.35. Für ein Ereignis $A \subseteq \Omega$ ist die zugehörige *Indikatorvariable* X_A definiert durch:

$$X_A(\omega) := \begin{cases} 1, & \text{falls } \omega \in A \\ 0, & \text{sonst.} \end{cases}$$

Für den Erwartungswert von X_A gilt: $\mathbb{E}[X_A] = \Pr[A]$.

Random Variables

Definition 2.39. Für eine Zufallsvariable X mit $\mu = \mathbb{E}[X]$ definieren wir die *Varianz* $\mathrm{Var}[X]$ durch

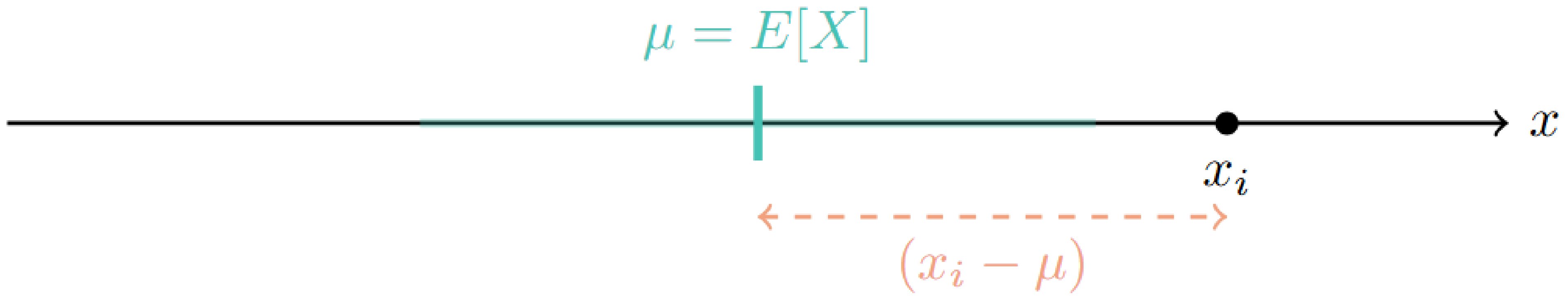
$$\mathrm{Var}[X] := \mathbb{E}[(X-\mu)^2] = \sum_{x \in W_X} (x-\mu)^2 \cdot \Pr[X = x].$$

Die Grösse $\sigma := \sqrt{\mathrm{Var}[X]}$ heisst *Standardabweichung* von X.

“the expected quadratic distance of X from its own expected value”

Random Variables

The variance $\text{Var}[X]$ represents the **expected quadratic distance** of X from its "mean" μ (expected value).



$$\text{Var}[X] = E[(X - \mu)^2]$$

Random Variables

Satz 2.40. Für eine beliebige Zufallsvariable X gilt

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Beweis. Sei $\mu := \mathbb{E}[X]$. Nach Definition gilt

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2 - 2\mu \cdot X + \mu^2].$$

Aus der Linearität des Erwartungswertes (Satz 2.33) folgt

$$\mathbb{E}[X^2 - 2\mu \cdot X + \mu^2] = \mathbb{E}[X^2] - 2\mu \cdot \mathbb{E}[X] + \mu^2.$$

Damit erhalten wir

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - 2\mu \cdot \mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \qquad \square$$

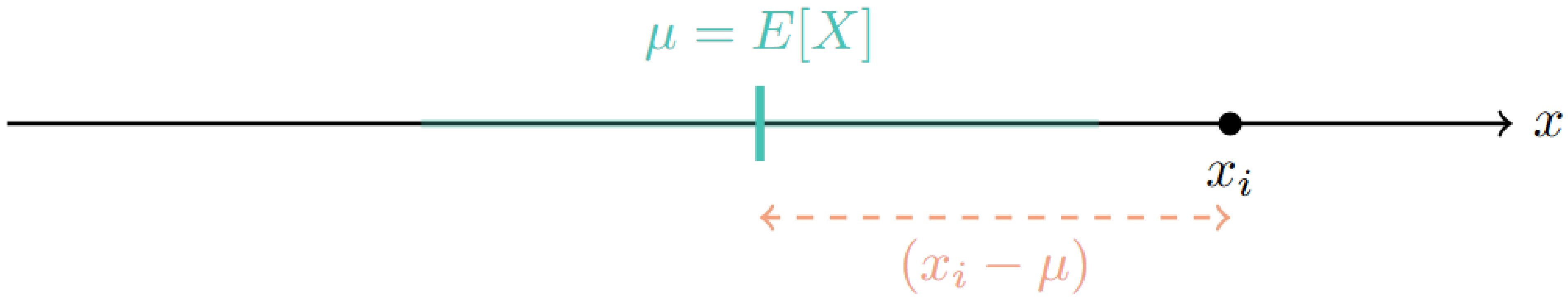
Random Variables

Satz 2.41. Für eine beliebige Zufallsvariable X und $a, b \in \mathbb{R}$ gilt

$$\mathrm{Var}[a \cdot X + b] = a^2 \cdot \mathrm{Var}[X].$$

Random Variables

The variance $\text{Var}[X]$ represents the **expected quadratic distance** of X from its "mean" μ (expected value).



$$\text{Var}[X] = E[(X - \mu)^2]$$

Distributions

Distributions

Important Distributions

Name	Notation	Support	Density	Expectation	Variance
Bernoulli	Bernoulli(p)	$\{0, 1\}$	$f_X(i) = \begin{cases} p & \text{for } i = 1, \\ 1-p & \text{for } i = 0. \end{cases}$	p	$p(1-p)$
Binomial	Bin(n, p)	$\{0, 1, \ldots, n\}$	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometric	Geo(p)	$\{1, 2, 3, \ldots\}$	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	$\{0, 1, 2, \ldots\}$	$f_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	λ	λ

2.5.1 Bernoulli-Verteilung

Eine Zufallsvariable X mit $W_X = \{0, 1\}$ und der Dichte

$$f_X(x) = \begin{cases} p & \text{für } x = 1, \\ 1 - p & \text{für } x = 0, \\ 0 & \text{sonst} \end{cases}$$

$\mathbb{E}[X] = p \quad \text{und} \quad \mathrm{Var}[X] = p(1-p)$

Distributions

We prove $\text{Var}[X] = p(1-p)$ for $X \sim \text{Bernoulli}(p)$ using the identity:

$$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X] = 1(p) + 0(1-p) = p \implies (\mathbb{E}[X])^2 = p^2$$

$$\mathbb{E}[X^2] = 1^2(p) + 0^2(1-p) = p$$

Plugging these into the variance formula:

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= p - p^2 \\ &= \mathbf{p(1 - p)}\end{aligned}$$

Distributions

2.5.2 Binomialverteilung

Eine Bernoulli-verteilte Zufallsvariable erhalten wir zum Beispiel als Indikator für 'Kopf', wenn wir ein Münze einmal werfen. Werfen wir die Münze statt dessen n-mal und fragen wie oft wir Kopf erhalten haben, so ist die entsprechende Zufallsvariable *binomialverteilt*.

"If I throw $n = 25$ coins, what is the probability that $i = 10$ land on heads"

Distributions

"If I throw n = 25 coins, what is the probability that i = 10 land on heads"

$$\Pr[X = i] = \sum_{\omega \in \{K,Z\}^n, X(\omega)=i} \Pr[\omega] = \sum_{\omega \in \{K,Z\}^n, X(\omega)=i} p^i (1-p)^{n-i}$$

$$= p^i (1-p)^{n-i} \cdot |\{\omega \in \{K,Z\}^n,\ \omega \text{ enthält genau } i\text{-mal Kopf}\}|,$$

"(Anzahl Abfolgen mit x-mal Kopf) *
P[Spezifische Abfolge {K,Z,...,K} trifft ein mit genau x-mal Kopf]"

→ $\binom{n}{x} p^x (1-p)^{n-x}$

Distributions

$$X \sim \mathrm{Bin}(n, p)$$

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, \ldots, n\} \\ 0, & \text{sonst.} \end{cases}$$

$$\mathbb{E}[X] = np \quad \text{und} \quad \mathrm{Var}[X] = np(1-p)$$