Week 6

Random Variables Exercise

Dynamic Programming CodeX Tutorial

Distributions

Multiple Random Variables

Infos

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Random Variables Recap

Random Variables

Definition 2.25. Eine *Zufallsvariable* ist ein Abbildung $X \colon \Omega \to \mathbb{R}$, wobei Ω die Ergebnismenge eines Wahrscheinlichkeitsraumes ist.

Bei diskreten Wahrscheinlichkeitsräumen ist der *Wertebereich* einer Zufallsvariablen

$$W_X := X(\Omega) = \{x \in \mathbb{R} \mid \exists\, \omega \in \Omega \text{ mit } X(\omega) = x\}$$

Random Variables

Every (discrete) random variable X can be naturally assigned two real-valued functions.

Dichte(funktion)
The function $f_X : \mathbb{R} \to [0,1]$ is defined as:

$$f_X(x) = \Pr[X = x]$$

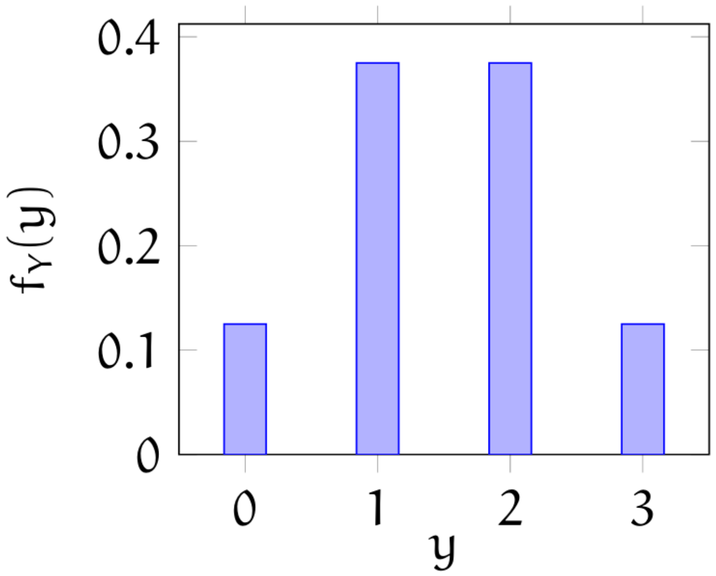
Verteilung(sfunktion)
The function $F_X : \mathbb{R} \to [0,1]$ is defined as:

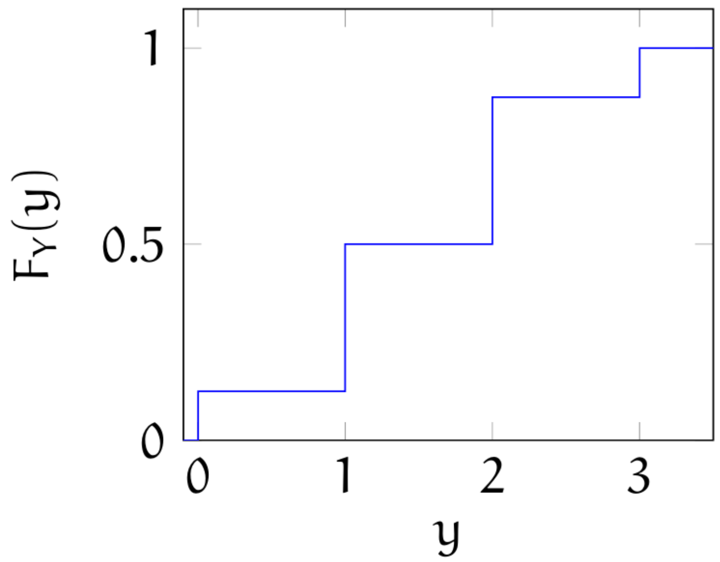
$$F_X(x) = \Pr[X \leq x] = \sum_{x' \in W_X : x' \leq x} \Pr[X = x']$$

Random Variables

Beispiel 2.26 *(Fortsetzung)* Für die Zufallsvariable Y erhalten wir

$$\begin{aligned}
\Pr[Y=0] &= \Pr[ZZZ] = \frac{1}{8},\\
\Pr[Y=1] &= \Pr[KZZ] + \Pr[ZKZ] + \Pr[ZZK] = \frac{3}{8},\\
\Pr[Y=2] &= \Pr[KKZ] + \Pr[KZK] + \Pr[ZKK] = \frac{3}{8},\\
\Pr[Y=3] &= \Pr[KKK] = \frac{1}{8}.
\end{aligned}$$





Expected Value and Variance Recap

(more details in last week’s slides)

Random Variables

Definition 2.27. Zu einer Zufallsvariablen X definieren wir den *Erwartungswert* $\mathbb{E}[X]$ durch

$$\mathbb{E}[X] := \sum_{x \in W_X} x \cdot \Pr[X = x],$$

sofern die Summe absolut konvergiert. Ansonsten sagen wir, dass der Erwartungswert undefiniert ist.

(only holds for **discrete** random variables)

Random Variables

Definition 2.27. Zu einer Zufallsvariablen X definieren wir den *Erwartungswert* $\mathbb{E}[X]$ durch

$$\mathbb{E}[X] := \sum_{x \in W_X} x \cdot \Pr[X = x],$$

sofern die Summe absolut konvergiert. Ansonsten sagen wir, dass der Erwartungswert undefiniert ist.

Lemma 2.29. Ist X eine Zufallsvariable, so gilt:

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega].$$

Random Variables

Satz 2.30. Sei X eine Zufallsvariable mit $W_X \subseteq \mathbb{N}_0$. Dann gilt

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \Pr[X \geq i].$$

Random Variables

1. The Standard Definition

By definition, the expected value is the weighted sum of all possible outcomes:

$$E[X] = \sum_{x=1}^{\infty} x \cdot \Pr[X = x] = 1 \cdot \Pr[X = 1] + 2 \cdot \Pr[X = 2] + 3 \cdot \Pr[X = 3] + \dots$$

2. Expanding the Tail Probabilities

The tail probability $\Pr[X \geq k]$ is the sum of all individual probabilities from k onwards. We can write these out row by row:

$$\begin{aligned}
\Pr[X \geq 1] &= \Pr[X = 1] + \Pr[X = 2] + \Pr[X = 3] + \Pr[X = 4] + \dots \\
\Pr[X \geq 2] &= \qquad\qquad\quad \Pr[X = 2] + \Pr[X = 3] + \Pr[X = 4] + \dots \\
\Pr[X \geq 3] &= \qquad\qquad\qquad\qquad\qquad\;\; \Pr[X = 3] + \Pr[X = 4] + \dots \\
\Pr[X \geq 4] &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Pr[X = 4] + \dots \\
&=
\end{aligned}$$

Random Variables

3. Vertical Summation

If we sum all these rows **vertically**, we count the occurrences of each $\Pr[X = x]$ term:

- $\Pr[X = 1]$ appears **1** time (only in the first row).
- $\Pr[X = 2]$ appears **2** times (in the first and second row).
- $\Pr[X = 3]$ appears **3** times (in the first, second, and third row).

Summing the columns gives us:

$$\sum_{k=1}^{\infty} \Pr[X \geq k] = 1 \cdot \Pr[X = 1] + 2 \cdot \Pr[X = 2] + 3 \cdot \Pr[X = 3] + \ldots = E[X]$$

Random Variables

Satz 2.33. *(Linearität des Erwartungswerts)* Für Zufallsvariablen $X_1, \ldots, X_n$ und $X := a_1 X_1 + \ldots + a_n X_n + b$ mit $a_1, \ldots, a_n, b \in \mathbb{R}$ gilt

$$\mathbb{E}[X] = a_1 \mathbb{E}[X_1] + \ldots + a_n \mathbb{E}[X_n] + b.$$

Random Variables

Definition 2.39. Für eine Zufallsvariable X mit $\mu = \mathbb{E}[X]$ definieren wir die *Varianz* $\mathrm{Var}[X]$ durch

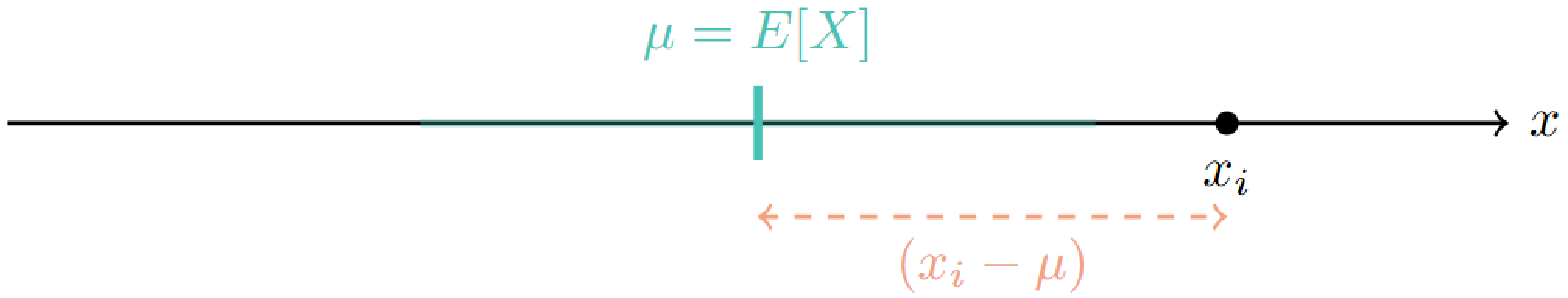
$$\mathrm{Var}[X] := \mathbb{E}[(X - \mu)^2] = \sum_{x \in W_X} (x - \mu)^2 \cdot \Pr[X = x].$$

Die Grösse $\sigma := \sqrt{\mathrm{Var}[X]}$ heisst *Standardabweichung* von X.

"the expected quadratic distance of X from its own expected value"

Random Variables

The variance Var$[X]$ represents the **expected quadratic distance** of X from its "mean" μ (expected value).



$$\mathrm{Var}[X] = E[(X-\mu)^2]$$

Random Variables

Satz 2.40. Für eine beliebige Zufallsvariable X gilt

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Beweis. Sei $\mu := \mathbb{E}[X]$. Nach Definition gilt

$$\mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu \cdot X + \mu^2].$$

Aus der Linearität des Erwartungswertes (Satz 2.33) folgt

$$\mathbb{E}[X^2 - 2\mu \cdot X + \mu^2] = \mathbb{E}[X^2] - 2\mu \cdot \mathbb{E}[X] + \mu^2.$$

Damit erhalten wir

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - 2\mu \cdot \mathbb{E}[X] + \mu^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2. \qquad \square$$

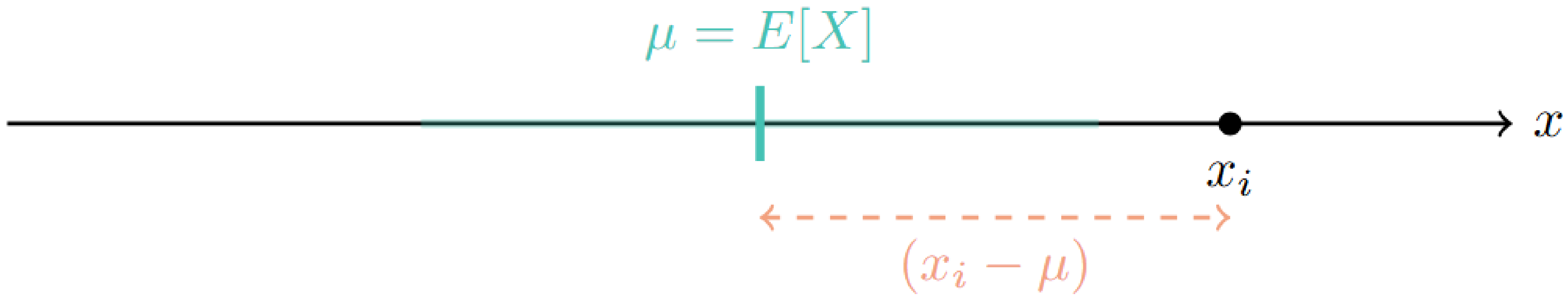
Random Variables

Satz 2.41. Für eine beliebige Zufallsvariable X und $a, b \in \mathbb{R}$ gilt

$$\mathrm{Var}[a \cdot X + b] = a^2 \cdot \mathrm{Var}[X].$$

Random Variables

The variance $\mathrm{Var}[X]$ represents the **expected quadratic distance** of X from its "mean" μ (expected value).



$$\mathrm{Var}[X] = E[(X - \mu)^2]$$

Exercise S7

Expected Value and Variance

Exercise S7.1 – *Intransitive Dice*

Alice, Bob, and Clara just found three fair dice. Instead of the numbers 1 to 6, the six sides of the dice show the following numbers:

Die 1: $2, 2, 4, 4, 9, 9$

Die 2: $1, 1, 6, 6, 8, 8$

Die 3: $3, 3, 5, 5, 7, 7$

Bob and Clara want to play a game: Bob starts by choosing a die. Next, Clara chooses a die. Then, both of them roll the dice and the person whose die shows the higher number wins.

Alice computes the expected value of the outcome of each die. She claims that their choices do not matter, because all dice have the same expected value.

Bob additionally computes the variance for the outcome of each die. He claims that the first two dice have a higher variance and are thus better than the third die. Thus, he chooses die 1.

Clara chooses die 3 and claims that her probability of winning was $\frac{5}{9}$ all along — independent of Bob's decision.

Compute the expected value and variance for each die. What do you think about the claims of Alice, Bob, and Clara?

DP CodeX Tutorial

Frog Feeding

Frog Feeding

There are n lotus leaves arranged in a straight line on the surface of a pond, each leaf having some number of flies around it. A frog jumps out of the pond onto one of the leaves and has the following plan of feeding itself.

First things first, it instantly eats all the f_i flies gathered around the leaf i on which the frog currently stands. After eating the flies, it jumps to the lotus leaf on the immediate left (leaf $i-1$) with probability p_i and otherwise, with probability $1-p_i$, to the lotus leaf on the immediate right. Everyone knows that frogs are not particularly intelligent creatures... Hence, if it jumps to the left of leaf 0 or to the right of leaf $n-1$, it ends up back in the pond and stops feeding.

However, mother nature is kind to the frog and does not want to let it starve. Consequently, as soon as the frog jumps away from the leaf i a new batch of f_i flies gather around it, ready to be eaten if the frog jumps on it again. In particular, {\em every time} the frog jumps on the lotus leaf i the number flies it eats is exactly f_i (for all $0 \leq i \leq n-1$).

After m jumps the frog realises its legs are just too tired of jumping around (unless it jumped into the pond even earlier). Thus, it stops feeding and goes back into the pond. What is the expected number of flies the frog will eat before returning to the pond?

Input

The first line of the input file contains a number $t \leq 30$ of test cases. Each of the t test cases is described as follows.

- It starts with a line that contains three integers `n k m`, separated by a space. They denote:
  - n, the number of lotus leaves ($0 \leq n \leq 10^3$);
  - k, the lotus leaf that the frog initially starts from ($0 \leq k \leq n-1$);
  - m, the (maximum) number of jumps the frog takes before it goes back into the pond ($0 \leq m \leq 10^2$).
- The following line contains n integers $\mathtt{f}_0$ ... $\mathtt{f}_{n-1}$, separated by a space, denoting the number of flies gathered around the lotus leaves ($0 \leq f_i \leq 10^2$, for all $i \in \{0, \ldots, n-1\}$).
- The following line contains n real numbers $\mathtt{p}_0$ ... $\mathtt{p}_{n-1}$, separated by a space, denoting the probability of the frog jumping to the left if it is currently on lotus leaf i ($0 \leq p_i \leq 1$, for all $i \in \{0, \ldots, n-1\}$).

Output

For each test case output a single line with the expected total number of flies the frog will eat before jumping back into the pond. Your solution is going to be accepted if it has an absolute or relative error of at most 10^{-3}.

Distributions

Distributions

Important Distributions

Name	Notation	Support	Density	Expectation	Variance
Bernoulli	Bernoulli(p)	$\{0, 1\}$	$f_X(i) = \begin{cases} p & \text{for } i = 1, \\ 1-p & \text{for } i = 0. \end{cases}$	p	$p(1-p)$
Binomial	Bin(n, p)	$\{0, 1, \ldots, n\}$	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometric	Geo(p)	$\{1, 2, 3, \ldots\}$	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	$\{0, 1, 2, \ldots\}$	$f_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	λ	λ

Bernoulli Distribution

2.5.1 Bernoulli-Verteilung

Eine Zufallsvariable X mit $W_X = \{0, 1\}$ und der Dichte

$$f_X(x) = \begin{cases} p & \text{für } x = 1, \\ 1 - p & \text{für } x = 0, \\ 0 & \text{sonst} \end{cases}$$

$\mathbb{E}[X] = p \quad \text{und} \quad \mathrm{Var}[X] = p(1 - p)$

Bernoulli Distribution

We prove $\mathrm{Var}[X] = p(1-p)$ for $X \sim \mathrm{Bernoulli}(p)$ using the identity:

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X] = 1(p) + 0(1-p) = p \implies (\mathbb{E}[X])^2 = p^2$$

$$\mathbb{E}[X^2] = 1^2(p) + 0^2(1-p) = p$$

Plugging these into the variance formula:

$$\begin{aligned}\mathrm{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= p - p^2 \\ &= \mathbf{p(1 - p)}\end{aligned}$$

Binomial Distribution

2.5.2 Binomialverteilung

Eine Bernoulli-verteilte Zufallsvariable erhalten wir zum Beispiel als Indikator für 'Kopf', wenn wir ein Münze einmal werfen. Werfen wir die Münze statt dessen n-mal und fragen wie oft wir Kopf erhalten haben, so ist die entsprechende Zufallsvariable *binomialverteilt*.

"If I throw $n = 25$ coins, what is the probability that $i = 10$ land on heads"

Binomial Distribution

"If I throw n = 25 coins, what is the probability that i = 10 land on heads"

$$\Pr[X = i] = \sum_{\omega \in \{K,Z\}^n, X(\omega)=i} \Pr[\omega] = \sum_{\omega \in \{K,Z\}^n, X(\omega)=i} p^i (1-p)^{n-i}$$

$$= p^i (1-p)^{n-i} \cdot |\{\omega \in \{K,Z\}^n,\ \omega \text{ enthält genau i-mal Kopf}\}|,$$

"(Anzahl Abfolgen mit x-mal Kopf) *
P[Spezifische Abfolge {K,Z,...,K} trifft ein mit genau x-mal Kopf]"

→ $\binom{n}{x} p^x (1-p)^{n-x}$

Binomial Distribution

$$X \sim \mathrm{Bin}(n, p)$$

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, \dots, n\} \\ 0, & \text{sonst.} \end{cases}$$

$$\mathbb{E}[X] = np \quad \text{und} \quad \mathrm{Var}[X] = np(1-p)$$

Poisson Distribution

"What is the probability that exactly $i = 10$ people suffer from a heart attack in an hour?"

- The expected number of heart attacks per hour is $\lambda = 5$ (not really)
- The number of time "instances" where a person could suffer a heart attack within an hour is essentially infinite $n \to \infty$ (informal)

Poisson Distribution

“What is the probability that exactly $i = 10$ people suffer from a heart attack in an hour?”

- The expected number of heart attacks per hour is $\lambda = 5$ (not really)
- The number of time “instances” where a person could suffer a heart attack within an hour is essentially infinite $n \to \infty$ (informal)

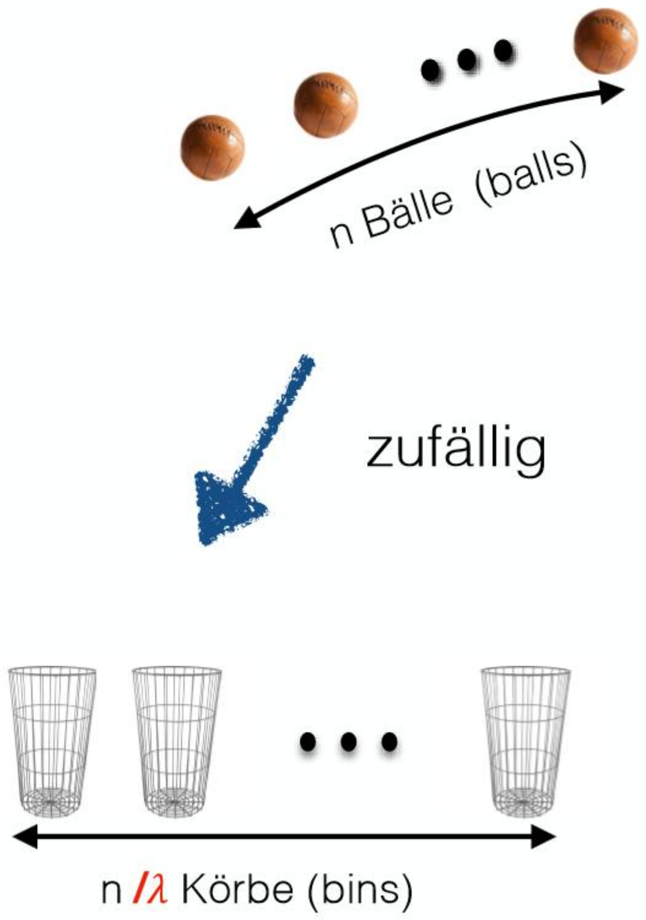
$$X \sim Po(\lambda)$$

$$f_X(i) = \begin{cases} \frac{e^{-\lambda}\lambda^i}{i!} & \text{für } i \in \mathbb{N}_0 \\ 0 & \text{sonst.} \end{cases}$$

$$\mathbb{E}[X] = \mathrm{Var}[X] = \lambda$$

Poisson Distribution

Bin(n,**λ**/n) konvergiert für n → ∞ gegen Po(**λ**)



Poisson Distribution

X := Anzahl Bälle im **ersten** Korb

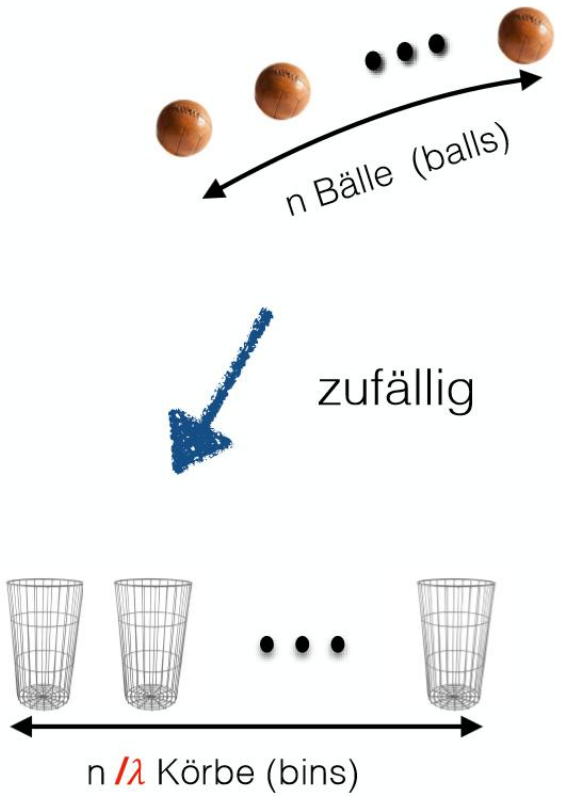
$X \sim \text{Bin}(n, \lambda/n)$

$$\mathbb{E}[X] = n \cdot \frac{\lambda}{n} = \lambda$$

$$\lim_{n\to\infty} \Pr[X = i] = \lim_{n\to\infty} \underbrace{\binom{n}{i}\left(\frac{\lambda}{n}\right)^i}\ \underbrace{\left(1 - \frac{\lambda}{n}\right)^{n-i}}$$

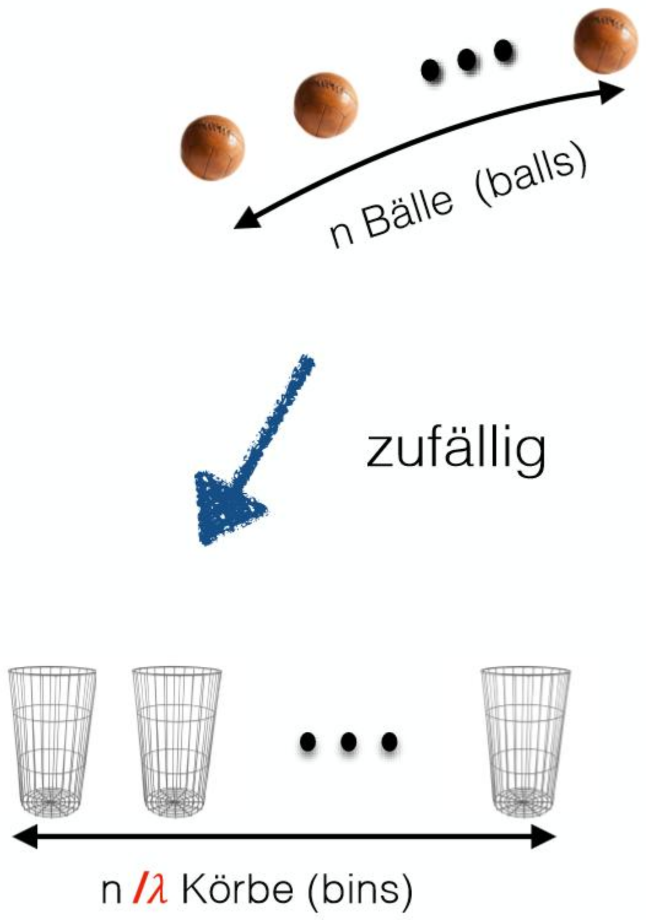
$$= \frac{n\cdot(n\text{-}1)\cdot\ldots\cdot(n\text{-}i\text{+}1)}{n\cdot n\cdot\ldots\cdot n}\cdot\frac{\lambda^i}{i!} \to \frac{\lambda^i}{i!} \qquad \approx (e^{-\frac{\lambda}{n}})^{n-i} = e^{-\lambda\frac{n-i}{n}} \to e^{-\lambda}$$

$$= \frac{\lambda^i}{i!}e^{-\lambda}$$



Poisson Distribution

Bin(n, λ/n) konvergiert für n $\rightarrow \infty$ gegen Po(λ)

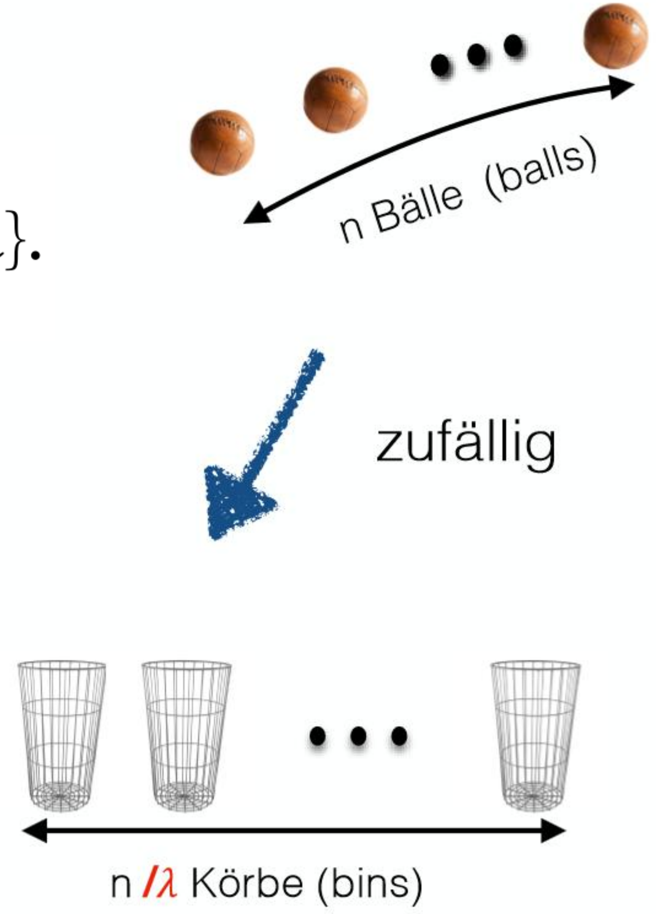


Poisson Distribution

- Case: λ = 1

$$X \sim \text{Bin}(n, 1/n)$$

$$f_X(i) = \binom{n}{i} \cdot \frac{1}{n^i} \cdot \left(1 - \frac{1}{n}\right)^{n-i} \qquad \text{für alle } i \in \{0, 1, \ldots, n\}.$$



Poisson Distribution

- Case: $\lambda = 1$

$$X \sim \mathrm{Bin}(n, 1/n)$$

$$f_X(i) = \binom{n}{i} \cdot \frac{1}{n^i} \cdot \left(1 - \frac{1}{n}\right)^{n-i} \qquad \text{für alle } i \in \{0, 1, \ldots, n\}.$$

$$\lim_{n \to \infty} f_X(i) = \lim_{n \to \infty} \frac{n(n-1)\ldots(n-i+1)}{i!} \cdot \frac{1}{n^i} \cdot \left(1 - \frac{1}{n}\right)^{n-i} = \frac{1}{i!} \cdot e^{-1},$$

das heisst, für $n \to \infty$ konvergiert X, bzw. die Binomialverteilung $\mathrm{Bin}(n, 1/n)$, gegen die Poisson-Verteilung $\mathrm{Po}(1)$.

Geometric Distribution

“I flip an unfair coin ($p = 0.4$) until it lands on heads. What is the probability that I need to wait exactly $i = 10$ flips until I succeed?”

Geometric Distribution

"I flip an unfair coin (p = 0.4) until it lands on heads. What is the probability that I need to wait exactly i = 10 flips until I succeed?"

$$X \sim \mathrm{Geo}(p)$$

$$f_X(i) = \begin{cases} p(1-p)^{i-1} & \text{für } i \in \mathbb{N} \\ 0 & \text{sonst.} \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{p} \quad \text{und} \quad \mathrm{Var}[X] = \frac{1-p}{p^2}$$

$$F_X(n) = \Pr[X \leq n] = \sum_{i=1}^{n} \Pr[X = i] = \sum_{i=1}^{n} p(1-p)^{i-1} = 1 - (1-p)^n$$

Geometric Distribution

Gedächtnislosigkeit.

Satz 2.45. Ist $X \sim \mathrm{Geo}(p)$, so gilt für alle $s, t \in \mathbb{N}$:

$$\Pr[X \geq s + t \mid X > s] = \Pr[X \geq t].$$

$$\Pr[X \geq s+t \mid X > s] = \frac{\Pr[X \geq s + t]}{\Pr[X > s]} = \frac{(1-p)^{s+t-1}}{(1-p)^s} = (1-p)^{t-1} = \Pr[X \geq t],$$

Geometric Distribution

Gedächtnislosigkeit.

Satz 2.45. Ist $X \sim \text{Geo}(p)$, so gilt für alle $s, t \in \mathbb{N}$:

$$\Pr[X \geq s + t \mid X > s] = \Pr[X \geq t].$$

Beweis. Wir wissen bereits, dass für die Verteilungsfunktion gilt: $F_X(n) = 1 - (1-p)^n$ für alle $n \in \mathbb{N}$. Somit gilt $\Pr[X \geq n] = (1-p)^{n-1}$ für alle $n \in \mathbb{N}$. Verwenden wir nun die Definition der bedingten Wahrscheinlichkeit, so erhalten wir daher

$$\Pr[X \geq s+t \mid X > s] = \frac{\Pr[X \geq s+t]}{\Pr[X > s]} = \frac{(1-p)^{s+t-1}}{(1-p)^s} = (1-p)^{t-1} = \Pr[X \geq t],$$

wie behauptet. □

Negative Binomial Distribution

The negative binomial distribution describes the probability of needing **exactly z independent trials** (each with success probability p) to achieve **exactly n successes**.

$$X \sim \mathrm{NB}(n, p)$$

"I flip an unfair coin ($p = 0.4$) until it has landed on heads $n = 10$ times. What is the probability I need to wait exactly $z = 20$ flips until I succeed?"

The **Dichte** $f_X(z)$ is defined for the number of trials $z \in \{n, n+1, n+2, \ldots\}$:

$$f_X(z) = \Pr[X = z] = \binom{z-1}{n-1} p^n (1-p)^{z-n}$$

Intuition: To have the n-th success at trial z, there must have been exactly $n-1$ successes in the previous $z-1$ trials.

Negative Binomial Distribution

Key Properties

The properties scale with the required number of successes n:

- **Expected Value:** $\mathbb{E}[X] = \frac{n}{p}$
- **Variance:** $\text{Var}[X] = \frac{n(1-p)}{p^2}$

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*Note: If $n = 1$, the distribution simplifies to the **Geometric Distribution** $X \sim Geo(p)$, representing the wait time for the first success.*

Distributions

Important Distributions

Name	Notation	Support	Density	Expectation	Variance
Bernoulli	Bernoulli(p)	$\{0,1\}$	$f_X(i) = \begin{cases} p & \text{for } i = 1, \\ 1-p & \text{for } i = 0. \end{cases}$	p	$p(1-p)$
Binomial	Bin(n,p)	$\{0,1,\ldots,n\}$	$f_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$	np	$np(1-p)$
Geometric	Geo(p)	$\{1,2,3,\ldots\}$	$f_X(i) = p(1-p)^{i-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Po(λ)	$\{0,1,2,\ldots\}$	$f_X(i) = \frac{e^{-\lambda}\lambda^i}{i!}$	λ	λ

Multiple Random Variables

Multiple Random Variables

Satz 2.58. Für zwei unabhängige Zufallsvariablen X und Y sei $Z := X + Y$. Es gilt

$$f_Z(z) = \sum_{x \in W_X} f_X(x) \cdot f_Y(z - x).$$

Multiple Random Variables

Satz 2.58. Für zwei unabhängige Zufallsvariablen X und Y sei $Z := X + Y$. Es gilt

$$f_Z(z) = \sum_{x \in W_X} f_X(x) \cdot f_Y(z - x).$$

Beweis. Mit Hilfe des Satzes von der totalen Wahrscheinlichkeit folgt, dass

$$\begin{aligned} f_Z(z) &= \Pr[Z = z] = \sum_{x \in W_X} \Pr[X + Y = z \mid X = x] \cdot \Pr[X = x] \\ &= \sum_{x \in W_X} \Pr[Y = z - x] \cdot \Pr[X = x] = \sum_{x \in W_X} f_X(x) \cdot f_Y(z - x). \end{aligned}$$

$\square$

Multiple Random Variables

Satz 2.60. *(Linearität des Erwartungswerts)* Für Zufallsvariablen $X_1, \dots, X_n$ und $X := a_1 X_1 + \dots + a_n X_n$ mit $a_1, \dots, a_n \in \mathbb{R}$ gilt

$$\mathbb{E}[X] = a_1 \mathbb{E}[X_1] + \dots + a_n \mathbb{E}[X_n].$$

Satz 2.61. *(Multiplikativität des Erwartungswerts)* Für unabhängige Zufallsvariablen $X_1, \dots, X_n$ gilt

$$\mathbb{E}[X_1 \cdot \dots \cdot X_n] = \mathbb{E}[X_1] \cdot \dots \cdot \mathbb{E}[X_n].$$

Satz 2.62. Für unabhängige Zufallsvariablen $X_1, \dots, X_n$ und $X := X_1 + \ldots + X_n$ gilt

$$\mathrm{Var}[X] = \mathrm{Var}[X_1] + \ldots + \mathrm{Var}[X_n].$$

Multiple Random Variables

Satz 2.65 (Waldsche Identität). N und X seien zwei unabhängige Zufallsvariable, wobei für den Wertebereich von N gilt: $W_N \subseteq \mathbb{N}$. Weiter sei

$$Z := \sum_{i=1}^{N} X_i,$$

wobei $X_1, X_2, \ldots$ unabhängige Kopien von X seien. Dann gilt:

$$\mathbb{E}[Z] = \mathbb{E}[N] \cdot \mathbb{E}[X].$$

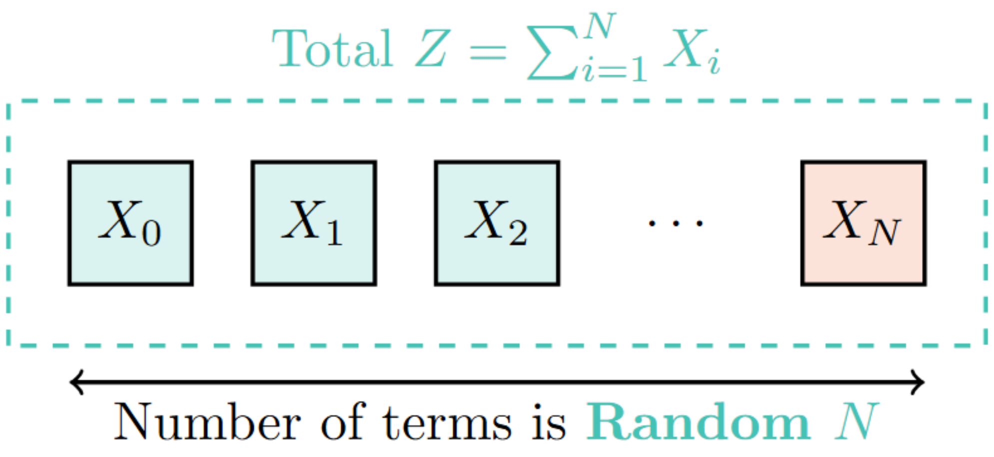
Multiple Random Variables

If N and X are **independent**, and Z is the sum of N independent copies of X, then:

$$\mathbb{E}[Z] = \mathbb{E}[N] \cdot \mathbb{E}[X]$$

Example

- **N (The Stopping Time):** A die roll deciding the number of bags of gummy bears we open. $\mathbb{E}[N] = 3.5$.
- **X (The Individual Value):** Red gummy bears per bag. Let's say $\mathbb{E}[X] = 10$.
- **Z (The Total):** The sum $X_1 + X_2 + \cdots + X_N$.



$$\mathbb{E}[Z] = \mathbb{E}[N] \cdot \mathbb{E}[X] = 3.5 \cdot 10 = 35$$